Instability of the ACW model, and problems with massive vectors during inflation

Burak Himmetoglu[(1)], Carlo R. Contaldi[(2)] and Marco Peloso[(1)]

[(1)] *School of Physics and Astronomy, University of Minnesota, Minneapolis, MN 55455, USA and*
[(2)] *Theoretical Physics, Blackett Laboratory, Imperial College, London, SW7 2BZ, UK*

We prove that the anisotropic inflationary background of the Ackerman-Carroll-Wise model, characterized by a fixed-norm vector field, is unstable. We found the instability by explicitly solving the linearized equations for the most general set of perturbations around this background, and by noticing that the solutions diverge close to horizon crossing. This happens because one perturbation becomes a ghost at that moment. A simplified computation, with only the perturbations of the vector field included, shows the same instability, clarifying the origin of the problem. We then discuss several other models, with a particular emphasis on the case of a nonminimal coupling to the curvature, in which vector fields are used either to support an anisotropic expansion, or to generate cosmological perturbations on an isotropic background. In many cases, the mass term of the vector needs to have the "wrong" sign; we show that, as a consequence, the longitudinal vector mode is a ghost (a field with negative kinetic term, and negative energy; not simply a tachyon). We comment on problems that arise at the quantum level. In particular, the presence of a ghost can be a serious difficulty for the UV completion that such models require in the sub-horizon regime.

I. INTRODUCTION

Cosmological observations over the past decade have ushered in the long heralded era of precision cosmology with theoretical models of the universe now being compared quantitatively to the data. In particular, Cosmic Microwave Background (CMB) anisotropies measurements from both orbital and sub-orbital experiments and Large Scale Structure (LSS) observations have revealed a universe in overall agreement with the standard inflationary paradigm. This picture assumes an epoch preceding the standard, radiation dominated era when the scale factor increased exponentially with a nearly constant Hubble rate. This quasi deSitter expansion not only provides a solution to the classical cosmological problems (homogeneity, isotropy, monopole and flatness) but also results in a nearly scale-invariant spectrum of super-horizon curvature perturbations which seed the structure formation process.

However, certain features of the full sky CMB maps recently observed by the WMAP satellite [1] seem to be in conflict with the standard picture. These so called 'anomalies' include the low power in the quadrupole moment [2, 3, 4], the alignment of the lowest multipoles, also known as the 'axis-of-evil' [5], and an asymmetry in power between the northern and southern ecliptic hemispheres [6]. The statistical significance of these effects has been debated extensively in the literature. For instance, the significance of the anomalous lack of large-angle correlations, together with the alignment of power has also grown in strength with the latest data [7], with only one in 4000 realizations of the concordance model in agreement with the observations.

Recent studies on properly masked data have shown that an anisotropic covariance matrix fits the WMAP low-ℓ data at the 3.8σ level [8]. [1] The fit was done using the full anisotropic covariance matrix and found a best-fit anisotropy at the 10% level. The increase in significance arises from the fact that much information is contained in the off-diagonal terms of the covariance matrix in these models [10, 11] and this is missed by simple fits to the diagonal power spectrum.

Such violations of statistical isotropy are considered at odds with the standard phase of early inflation. However, an albeit more plausible explanation of these anomalies arising from a systematic effect or foreground signal affecting the analysis is not forthcoming. This has led to a number of attempts at reconciling some of the anomalies with the standard inflationary picture through various modifications.

It has been suggested in [10, 11] that the alignment of lowest multipoles could be could be due to an early anisotropic expansion. Indeed, an anisotropic expansion results in a nonvanishing correlation between different $a_{\ell m}$ coefficients of the CMB multipole expansion (the formalism for dealing with the cosmological perturbations on an anisotropic background was also developed in [12, 13]). To realize this with a minimal departure from the standard inflationary picture, the anisotropy was simply set as an initial condition at the onset of inflation. It was then shown in [14] that one of the gravity waves polarization experiences a large growth during the anisotropic era (this is intimately related to the instability of Kasner spaces [14]), which may result in a large B signal in the CMB. Inflation however rapidly removes the anisotropy. The modes that leave the horizon well after the universe has isotropized were deep inside

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[1] An upper bound on the anisotropy has instead been obtained in Ref. [9].

the horizon while the universe was anisotropic, and one recovers a standard power spectrum at the corresponding scales. The signature of the earlier anisotropic stage are therefore present only on relatively large scales, which were comparable to the horizon when the universe was anisotropic. If the duration of inflation was too long, such scales get inflated well beyond the present horizon, and are irrelevant for phenomenology. Therefore, the above suggestions require a tuned duration of inflation.

To avoid this tuning, one can obtain prolonged anisotropic inflationary solutions by introducing some ingredients that violate the premises of Wald's theorem [15] on the rapid isotropization of Bianchi universes. This has been realized through the addition of quadratic curvature invariants to the gravity action [16], with the use of the Kalb-Ramond axion [17], or of vector fields [18]. [2] The basic idea underlying these models is that one or more vector fields acquire a spatial expectation value (vev). The background solutions are homogeneous, so that the vev is only time dependent, and aligned along one direction; for definiteness, we take it to coincide with the $x-$axis. The models differ from the way in which the vector field(s) acquires its vev. Most of this work we focus on one of the above models: namely the model by Ackerman, Carroll, and Wise (ACW) [21], in which the vector acquires a fixed norm due to the constraint equation enforced by a lagrange multiplier. There is also a cosmological constant in the model, which is responsible for the accelerated expansion. The vev of the vector field controls the difference between the expansion rate of the x-direction and that of the two orthogonal directions.

The original paper studied the power spectrum generated for a test field $\delta\chi$ in this background. It was then argued that this spectrum gets transferred to the spectrum of primordial density perturbations through the mechanism of modulated perturbations [22]. The basic idea is as follows: imagine that the effective cosmological constant in the model is replaced by an inflaton field, and that the decay rate of the inflaton is controlled by the vacuum expectation value of the test field χ. Then, fluctuations $\delta\chi$ will result in fluctuations of the decay rate. This generates fluctuations of the energy density of the decay products (since the energy density of the inflaton and of the decay products scale differently with time). The resulting power spectrum was the starting point of the analysis of Ref. [8] cited above. The claim of [8] renders the ACW model particularly interesting. However, the mechanism of modulated perturbations requires that the perturbations of the metric and of the energy densities which are unavoidably generated during inflation are subdominant; otherwise the resulting density perturbations will be a superposition of the original ones, generated without the test field, plus those imprinted by $\delta\chi$ at reheating. The original ACW work computed the perturbations of χ on an unperturbed background. The underlying assumption is that the original perturbations in the metric and in the energy densities can be neglected with respect to $\delta\chi$. In this work we show that, unfortunately, this assumption does not hold. On the contrary, each mode (in Fourier space) of the linearized perturbations (which are unavoidably generated irrespectively of the test field χ) becomes nonlinear while it leaves the horizon during inflation. As a consequence, the background solution of [21] is unstable, and any phenomenological study based on it is unreliable.

The computation of the instability is rather complicated, due to the large number of perturbations involved. An important step in this computation was performed in Ref. [23], [3] where the linearized equations for the complete set of perturbations of the ACW model were presented. These equations were solved in [23] in the approximation of very large or very small wavelength of the perturbations. Since the Hubble expansion rates for the ACW model are constant, the wavelength of any mode increases exponentially with time. Therefore, at sufficiently early times during inflation, the wavelength is much smaller than the inverse hubble rates; this is the small wavelength regime. On the contrary, at sufficiently late times during inflation, the wavelength is much greater than the inverse hubble rates; this is the large wavelength regime. The moment at which those length become equal is known as horizon crossing [4]. We show here that the linearized perturbations diverge close to horizon crossing, where the equations were not solved in [23] (apart from this, our computation agrees with that of [23]). This computation is presented in Section II. It is just a conceptually straightforward (although technically involved) computation and explicit solution of the equations for the complete linearized system of perturbations for the ACW model. Each mode of the perturbations becomes unstable at a time t_* close to horizon crossing (since different modes have different wavelengths, they all become unstable at different times). This analysis is all one needs to prove that the ACW background solution is unstable.

To understand the reason for the instability, in Section III we compute the quadratic action for the perturbations.

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[2] The possibility of a late time anisotropic expansion has instead been considered in [19]. Although the present statistics of the observed Supernovae does not show any evidence for the anisotropy [20], such studies are motivated by the large increase of data that is expected in the next few years.

[3] Another interesting analysis can be found in Ref. [24], where the late time stability of the ACW model was studied, assuming that the cosmological constant is replaced by a fluid.

[4] The language is slightly loose, since there are two different expansion rates for the $x-$direction, and for the perpendicular $y-z$ plane. However, we have in mind a small anisotropy (since a large one is forbidden by observations), so that the two rate are parametrically identical.

After the action is diagonalized, we find that the kinetic term for one of the modes goes from positive to negative precisely at the time t_*. This has two consequences for the perturbations: 1) the equations of motion for the perturbations diverge at this point, as we have seen; 2) the mode is a ghost for $t > t_*$ (we note that this second point alone renders the model unstable, due to the decay of the vacuum in ghost-nonghost pairs, with a rate that increases with the momentum of the produced particles). This last point is very interesting in the light of the findings of Ref. [25]. It was shown there that, in models in which the norm of the vector field is fixed as in the ACW model, it is possible to construct field configurations with arbitrary large negative energy. Those configurations were obtained ignoring gravity. However, combining the result of Ref. [25] and ours, we are inclined to conclude that such models are generally unstable due to ghosts.

In Section IV we present a simplified computation, with only the perturbations of the vector field included (no metric perturbations). This is by itself an inconsistent computation, since metric perturbations are unavoidably sourced by the perturbations of the vector field. However, we show it for illustrative purposes, since it is much simpler than the complete computations presented in the first two parts. The qualitative results of this simplified computation are in perfect agreement with those of the complete one (the solutions diverge close to horizon crossing, when one mode becomes a ghost). This can be seen with extremely simple algebra: the mode that becomes a ghost is the longitudinal perturbation of the vector field ("longitudinal" in the $y - z$ plane; see Section IV for the technical detail). Including the perturbations of the metric (as was done in the first two parts of this paper) simply "dresses up" the problem with technical complication, but does not remove the instability of the model, which is ultimately related to the vector field.

The instability found for the ACW model motivated us to study also different models in which vector fields with broken U(1) invariance play a role during inflation. In Section V, we specifically focus on models in which the mass term is due to a nonminimal coupling of the vector to the curvature [26, 27, 28]. These models also present a ghost. This is not related to an anisotropic expansion (in fact, the background is isotropic for [26], in which the vector fields do not contribute to the expansion, and and can be made isotropic for [27] by using $3N$ mutually orthogonal vectors with equal vev), but rather to the specific sign that the mass term needs to have in these models. This is due to the fact that, also in this case, the ill-behaved mode is the longitudinal polarization of the vector, which is present (and, therefore, acquires a kinetic term) only when the mass is nonvanishing. Since this is not a trivial point, as since this has not yet been pointed out for the above models, we present three different proofs of this statement. In addition to the ghost, we have verified that also for the models [27, 28] the linearized equations for the perturbations diverge close to the horizon crossing, precisely as in the ACW model. The detailed computations will be presented elsewhere.

In Section VI we present some general discussions of these models. We do not provide quantitative computations here, as in the rest of the paper, but we present some comments and some worries that we believe deserve consideration. Particularly, we comment on some of the implications of having a ghost state in the theory. In the above models, the mass term provides a hard breaking of the gauge invariance. The interactions of the longitudinal vector polarization become strong at an energy scale parametrically set by the mass of the vector. This will occur for all the interactions involving the longitudinal vector, or mediated by it, renormalizing the values of the couplings between all the fields in the theory. Since the mass in these models is the hubble rate, or below, this may invalidate the predictions that rely on the initial quantization in the short wavelength regime (depending on the precise point in which the theory goes out of control). The most immediate solution to this problem is to provide a UV completion of the theory, in which the gauge symmetry is broken spontaneously from the condensation of a field that becomes dynamical above that mass scale. However, providing a UV completion appears to be a harder task when the longitudinal vector is a ghost rather than a normal field (for instance, in the higgs mechanism, the field responsible for the mass would need to be itself a ghost). We believe that such issues deserve further study.

Most of the computations presented here we summarized in our previous work [29].

II. INSTABILITY FROM THE LINEARIZED EQUATIONS

In this Section we write down and solve the linearized equations for the most general set of perturbations of the ACW background solution. The model, and the background solution, are presented in Subsection II A. In Subsection II B we introduce the perturbations; we show how to classify them in two distinct sets that are decoupled from each other at the linearized level, and how to construct gauge invariant variables. In Subsection II C we write down and solve the linearized equations for the gauge invariant combinations. Some intermediate steps of this computation are put in Appendix A.

A. The ACW model

The ACW model is defined by the action,

$$S = \int d^4x \sqrt{-g} \left[\frac{M_p^2}{2} R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \lambda \left(A^2 - m^2 \right) - V_0 \right] \tag{1}$$

where $A^2 \equiv A_\mu A^\mu$ and V_0 is a constant vacuum energy (which is assumed to approximate a slow-roll inflationary phase). This action is a special case of the one considered in [21], which contains generalized kinetic terms. [5]

The evolution of the system is governed by the field equations derived from the action (1):

$$G_{\mu\nu} = \frac{1}{M_p^2} \left[F_{\mu\alpha} F_\nu{}^\alpha - 2\lambda A_\mu A_\nu + g_{\mu\nu} \left(-\frac{1}{4} F^2 + \lambda \left(A^2 - m^2 \right) - V_0 \right) \right]$$

$$\frac{1}{\sqrt{-g}} \partial_\nu \left[\sqrt{-g}\, F^{\mu\nu} \right] = 2\,\lambda\, A^\mu$$

$$A^2 = m^2 \tag{2}$$

which are, respectively, the Einstein equations, the equation of the vector field, and the constraint forced by the lagrange multiplier. The constraint enforces a fixed norm of the vector.

The background metric and vector field are

$$\langle g_{\mu\nu} \rangle = \text{diag} \left(-1,\, a\,(t)^2,\, b\,(t)^2,\, b\,(t)^2 \right)$$

$$\langle A_\mu \rangle = \left(0,\, M_p\, a\,(t)\, B_1\,(t),\, 0,\, 0 \right) \tag{3}$$

where in the vev for the vector field $\langle A_1 \rangle \equiv M_p\, a\, B_1$ we rescaled out the scale factor a and the reduced Planck mass M_p for algebraic convenience (since the background equations of motion are simpler when written in terms of B_1; we also note that B_1 is a dimensionless quantity). The last of (2) enforces a constant value for B_1,

$$B_1 = \frac{m}{M_p} \equiv \mu \tag{4}$$

Denoting the two expansion rates $H_a \equiv \dot{a}/a$ and $H_b \equiv \dot{b}/b$, where dot is a time derivative, the nontrivial equations in (2) are [6]

$$2H_a H_b + H_b^2 = \frac{1}{2} H_a^2 \mu^2 + \frac{V_0}{M_p^2}$$

$$2\dot{H}_b + 3H_b^2 = \frac{1}{2} H_a^2 \mu^2 + \frac{V_0}{M_p^2} + 2\lambda\mu^2$$

$$\dot{H}_a + \dot{H}_b + H_a^2 + H_a H_b + H_b^2 = -\frac{1}{2} H_a^2 \mu^2 + \frac{V_0}{M_p^2}$$

$$\lambda = H_a H_b + \frac{1}{2} \dot{H}_a \tag{5}$$

The equations are solved by exponentially expanding scale factors a and b, with constant Hubble rates

$$H_a^2 = \frac{2\,V_0}{M_p^2} \frac{1}{6 + 7\mu^2 + 2\mu^4} \;\;,\;\; H_b = \left(1 + \mu^2 \right) H_a \tag{6}$$

The rate H_b which characterizes the expansion of the two dimensional $y - z$ plane is larger than H_a, the expansion rate of the anisotropic x-direction. The difference is proportional to the background expectation value of the vector field. The overall inflationary expansion is also supported by the vacuum energy $V_0 > 0$.

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[5] Specifically, the general kinetic term for the vector field considered in [21] is $\mathcal{L} = -\beta_1 \nabla^\mu A^\nu \nabla_\mu A_\nu - \beta_2 \left(\nabla_\mu A^\mu \right)^2 - \beta_3 \nabla^\mu A^\nu \nabla_\nu A_\mu$. Ref. [23] showed that the cases $\beta_1 < 0$ and $\beta_1 + \beta_2 + \beta_3 \neq 0$ have ghosts. We have restricted our attention to the case of a standard kinetic term, with $\beta_1 = -\beta_3 = 1/2$, $\beta_2 = 0$.

[6] Specifically, eqs. (5) are, respectively, the 00, 11, 22 Einstein equations, and the $x-$ component of the vector field equation.

B. Classification of the perturbations and gauge invariant combinations

We generalize to the present case the standard computation of cosmological perturbations in the case of a scalar field on an isotropic background (see [30] for a review). It is convenient to exploit the symmetry of the background in this computation. For instance, in the isotropic case it is customary to classify the perturbations in scalar / vector / tensor modes, according to how they transform under spatial rotations. The underlying reason is that modes that transform differently are decoupled from each other at the linearized level (namely, in the linearized equations for the perturbations, or, equivalently, in the quadratic expansion of the action in the perturbations). Since the modes in that decomposition form a complete basis for the perturbations, we could still use them here. However, this would be of no practical advantage, since modes belonging to different representations would now be coupled to each other. We can however still exploit the residual symmetry of the ACW background solution (3) in the $y - z$ plane, and classify the perturbations according to how they transform with respect to rotations in this plane [11]. Modes transforming differently under these rotations are decoupled at the linearized level (we explicitly verify this in the equations of Subsection II C).

Specifically, we write the most general perturbations of the metric and of the vector field as

$$
\delta g_{\mu\nu} \;=\; \begin{pmatrix} -2\Phi & a\,\partial_1\,\chi & b\,(\partial_i\,B + B_i) \\[4pt] & -2\,a^2\,\Psi & a\,b\,\partial_1\left(\partial_i\,\tilde{B} + \tilde{B}_i\right) \\[4pt] & & b^2\left(-2\Sigma\,\delta_{ij} - 2\,\partial_i\,\partial_j\,E - \partial_i\,E_i - \partial_j\,E_i\right) \end{pmatrix}
$$

$$
\delta A_\mu \;=\; (\alpha_0,\,\alpha_1,\,\partial_i\alpha + \alpha_i) \tag{7}
$$

where $i, j = 1, 2$ span the isotropic coordinates. [7] The perturbations $\{\Phi,\,\chi,\,B,\,\Psi,\,\tilde{B},\,\Sigma,\,E,\,\alpha_0,\,\alpha_1,\,\alpha\}$ are 2d scalar modes: they encode 1 degree of freedom (d.o.f.) each. The perturbations $\{B_i,\,\tilde{B}_i,\,E_i,\,\alpha_i\}$ are 2d vector modes. Due to the transversality condition ($\partial_i B_i = \cdots = 0$), they also encode 1 d.o.f. each. Notice that, contrary to the 3d case, there are no 2d tensor modes, since the transversality and traceless conditions eliminate all possible degrees of freedom. Altogether, we have 10 d.o.f. in the 2d scalar sector, and 4 d.o.f. in the 2d vector sector. These add up to 14 d.o.f., which is the number of initial independent entries in the metric and in the vector field.

The set of perturbations just given is redundant, since modes can be transformed into each other through infinitesimal coordinate transformations $x^\mu \to x^\mu + \xi^\mu$. Under this change [30]

$$
\delta g_{\mu\nu} \to \delta g_{\mu\nu} - \langle g_{\mu\nu,\alpha}\rangle\xi^\alpha - \langle g_{\mu\alpha}\rangle\xi^\alpha_{,\nu} - \langle g_{\alpha\nu}\rangle\xi^\alpha_{,\mu} \quad,\quad \delta A_\mu \to \delta A_\mu - \langle A_{\mu,\alpha}\rangle\xi^\alpha - \langle A_\alpha\rangle\xi^\alpha_{,\mu} \tag{8}
$$

We decompose also the components of the infinitesimal parameter ξ^μ in the $y - z$ plane in a 2d scalar plus 2d vector part. In this way 2d scalar (vector) modes manifestly transform into 2d scalar (vector) modes.

To eliminate the redundancy, we can specify a gauge that completely fixes the freedom of general coordinate reparametrization (this was the procedure chosen in [11], where the choice $E = \Sigma = \tilde{B} = E_i = 0$ was made). Equivalently, we can construct gauge invariant combinations of the above perturbations ("gauge invariance" here means invariance with respect to general coordinate transformations; there is no gauge U(1) symmetry associated to the vector field, due to its mass term). We choose this second procedure here: each linearized equation can be written as "left hand side = 0"; therefore the equation must be gauge invariant. This means that the perturbations must enter in the linearized equations in such a way that these equations can be written in terms of the gauge invariant combinations only (the same is true for the quadratic action that we compute in the next Section). This provides a nontrivial check on our algebra.

To find the gauge invariant combinations, we compute how each mode in (7) transforms under (8), and we then

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[7] We chose to insert a ∂_1 derivative in the δg_{01} and δg_{1i} metric perturbations for algebraic convenience. For instance, we could have equivalently denoted $\delta g_{01} = \chi_1$. More precisely, we only study the modes of the perturbations which have a nonvanishing momentum component both along the x direction and in the $y - z$ plane (in coordinate space, they have a nontrivial dependence both on x and at least on one between y and z).

arrange them into invariant combinations. We find the following set of gauge invariant modes

$$\hat{\Phi} = M_p \left[\Phi + \left(\frac{\Sigma}{H_b} \right)^{\bullet} \right]$$

$$\hat{\Psi} = M_p \left[\Psi - \frac{H_a}{H_b} \Sigma + \frac{b}{a} \partial_1^2 \left(\tilde{B} + \frac{b}{a} E \right) \right]$$

$$\hat{B} = -\frac{M_p^2}{b} \vec{\partial}_T^2 \left[B - \frac{1}{b\,H_b} \Sigma + b\,\dot{E} \right]$$

$$\hat{\chi} = -\frac{M_p}{a} \partial_1^2 \left[\chi - \frac{1}{a\,H_b} \Sigma - a \left(\frac{b}{a} \left(\tilde{B} + \frac{b}{a} E \right) \right)^{\bullet} \right]$$

$$\hat{\alpha}_1 = -\frac{1}{a} \left[\alpha_1 + a\,M_p \frac{\dot{B}_1 + H_a\,B_1}{H_b} \Sigma - b\,M_p\,B_1\,\partial_1^2 \left(\tilde{B} + \frac{b}{a} E \right) \right]$$

$$\alpha = -\frac{1}{a} \partial_1 \left[\alpha - b\,M_p\,\partial_1 \left(\tilde{B} + \frac{b}{a} E \right) \right]$$

$$\hat{\alpha}_0 = \frac{1}{a} \partial_1 \left[\alpha_0 - a\,M_p\,B_1\,\partial_1 \left(\frac{b}{a} \left(\tilde{B} + \frac{b}{a} E \right) \right)^{\bullet} \right] \tag{9}$$

(where the bullet denotes time derivative, and $\vec{\partial}_T^2 \equiv \partial_2^2 + \partial_3^2$) in the 2d scalar sector, and

$$\hat{B}_i = B_i + b\,\dot{E}_i$$

$$\hat{\tilde{B}}_i = a \left(\tilde{B}_i + \frac{b}{a} E_i \right)$$

$$\hat{\alpha}_i = \frac{\alpha_i}{M_p} \tag{10}$$

in the 2d vector sector. The prefactors in front of these modes do not affect their gauge invariance, but have been chosen for algebraic convenience.

Since we started from ten 2d scalars and four 2d vectors, and since there are three infinitesimal 2d scalar transformations, and one infinitesimal 2d vector transformation, we end up with seven 2d scalar gauge invariant modes, and three 2d vector gauge invariant modes. One could have equivalently chosen other gauge invariant combinations of the initial modes. However, they can be obtained from those given here. [8]

We perform the computations in momentum space:

$$\delta(x) = \frac{1}{(2\pi)^3} \int d^3k\,\delta(k)\,\mathrm{e}^{-i\vec{k}\,\vec{x}} \tag{11}$$

where δ denotes any of the above perturbations, and where we denote with the same symbol the perturbation both in coordinate and in momentum space. k is the comoving momentum of the mode $\delta(k)$. Different modes (characterized each by a specific comoving momentum) are decoupled from each other at the linearized level. We denote by k_L the component of the comoving momentum in the $x-$direction, and by k_T the component in the orthogonal $y-z$ plane. The corresponding components of the physical momentum are $p_L = k_L/a(t)$ and $p_T = k_T/b(t)$. The square magnitude of the comoving/physical momenta are given by $k^2 = k_L^2 + k_T^2$ and $p^2 = p_L^2 + p_T^2$, respectively. Finally k_{Ti} (p_{Ti}) denotes the component of the transverse comoving (physical) momentum in the y ($i=2$) or z ($i=3$) direction.

In momentum space, the transversality conditions on the 2d vectors reads

$$k_{Ti}v_i = p_{Ti}v_i = 0 \tag{12}$$

where v_i denotes any of the 2d vector perturbations.

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[8] One can verify that the present choice of variables leads to the same results as the standard computation of [30], in the case of isotropic background, and of a scalar field instead of the vector. To do so, consider the gauge specified in [11] (in practice, it amounts in identifying each "hatted" mode with the corresponding "non-hatted" one). It was explicitly verified in [11] that, in the isotropic limit, this gauge leads to the same results as the longitudinal gauge. The perturbations in the longitudinal gauge can be immediately "promoted" to the gauge invariant modes used in [30]. This explicitly verifies that our formalism reproduces the standard one in the isotropic limit.

C. Solutions to the linearized equations for the perturbations

We perturb the metric and the vector field as in eq. (7). We want to solve the system of linearized equations for these modes. We multiply the second line of (2) by A_μ and we replace A^2 by m^2 on the right hand side. In this way we obtain an expression for λ in terms of the vector field and the metric (this is the procedure also adopted in [31] to study a similar model, and it has the advantage that we do not have to introduce explicitly the perturbation of the lagrange multiplier). In this way, the first and second line of (2) become

$$G_{\mu\nu} = \frac{1}{M_p^2} T_{\mu\nu} = \frac{1}{M_p^2} \left[F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{\mu^2 M_p^2} A_\alpha \nabla_\beta F^{\alpha\beta} A_\mu A_\nu + g_{\mu\nu} \left(-\frac{1}{4} F^2 - V_0 \right) \right]$$

$$\partial_\nu \left[\sqrt{-g}\, F^{\lambda\nu} \right] \left(m^2 \delta_\lambda^\mu - A_\lambda A^\mu \right) = 0 \tag{13}$$

We then linearize these equations, and we go to momentum space as indicated in (11). We denote the resulting equations as [9]

$$\mathrm{Eq}_{\mu\nu} : \delta \left(G_{\mu\nu} - \frac{1}{M_p^2} T_{\mu\nu} \right) = 0 \quad , \quad \mathrm{Eq}_\mu : \delta \left(\frac{\delta S}{\delta A_\mu} \right) = 0 \tag{14}$$

The explicit expressions are given in eqs. (A1) in Appendix A. The expressions pass two crucial tests: we initially wrote them in terms of the original perturbations (7); with some algebra, we have been able to write them solely in terms of the gauge invariant combinations (9) and (10). As a second test, we show explicitly in Appendix A that the set of 2d scalar and 2d vector perturbations decouple in these equations. We verified that the system of 2d vectors is stable. For brevity, we do not report those computations here. We also show in Appendix A that not all the equations (14) are independent (due to the perturbed Bianchi identities). A set of independent equations is

$$\mathrm{Eq}_1 : \quad \hat\alpha_1 - \mu\, \hat\Psi = 0 \tag{15}$$

$$\mathrm{Eq}_{00} : \quad \left(2 + \mu^2\right) H_a\, \dot{\hat\Psi} + p_T^2\, \hat\Psi + \left(2 + \mu^2\right) \left(3 + 2\mu^2\right) H_a^2\, \hat\Phi$$
$$-2\left(1 + \mu^2\right) H_a\, \hat\chi - \left(2 + \mu^2\right) H_a\, \hat B - \mu\, H_a\, \hat\alpha_0 = 0 \tag{16}$$

$$\mathrm{Eq}_{01} : \quad 2\left(1 + \mu^2\right) H_a \hat\Phi + \frac{1}{2\, p_L^2} \left[p_T^2 - 4\,\mu^2 \left(1 + \mu^2\right) H_a^2 \right] \hat\chi - \frac{\hat B}{2} - 2\mu\left(1 + \mu^2\right) \frac{H_a^2}{p_L^2} \hat\alpha_0 = 0 \tag{17}$$

$$\mathrm{Eq}_{0i} : \quad \dot{\hat\Psi} + \left(2 + \mu^2\right) H_a\, \hat\Phi - \frac{\hat\chi}{2} + \frac{p_L^2}{2 p_T^2}\, \hat B - \mu\, H_a\, \hat\alpha = 0 \tag{18}$$

$$\mathrm{Eq}_{11} : \quad \mu^2\, \ddot{\hat\Psi} + \mu^2 \left(3 + 2\mu^2\right) H_a\, \dot{\hat\Psi} + \mu^2\, p_T^2\, \hat\Psi - \left(2 + \mu^2\right) H_a\, \dot{\hat\Phi} + \left[p_T^2 - \left(2 + \mu^2\right)\left(3 + 2\mu^2\right) H_a^2 \right] \hat\Phi$$
$$+ \dot{\hat B} + \left(3 + 2\,\mu^2\right) H_a\, \hat B + \mu\, \dot{\hat\alpha}_0 + \mu\left(3 + 2\,\mu^2\right) H_a\, \hat\alpha_0 - \mu\, p_T^2\, \hat\alpha = 0 \tag{19}$$

$$\mathrm{Eq}_0 : \quad \dot{\hat\alpha} + H_a\, \hat\alpha + \frac{1}{p_T^2} \left[p^2 - 2\left(1 + \mu^2\right) H_a^2 \right] \hat\alpha_0 + \mu\, \frac{p_L^2}{p_T^2}\, \dot{\hat\Psi} + \mu\, \frac{p_L^2}{p_T^2}\, H_a\, \hat\Phi - \mu\left(1 + \mu^2\right) \frac{2 H_a^2}{p_T^2}\, \hat\chi = 0 \tag{20}$$

$$\mathrm{Eq}_i : \quad \ddot{\hat\alpha} + 3 H_a\, \dot{\hat\alpha} + \dot{\hat\alpha}_0 + \left(p_L^2 - 2\,\mu^2\, H_a^2 \right) \hat\alpha - \mu\, p_L^2\, \hat\Psi + \mu\, \frac{p_L^2}{p_T^2}\, H_a\, \hat B + 2 H_a\, \hat\alpha_0 = 0 \tag{21}$$

where we remind that p_L and p_T denote, respectively, the component of the physical momentum of the perturbations along the x direction and in the orthogonal $y - z$ plane. We have used (15) to eliminate $\hat\alpha_1$ in favor of $\hat\Psi$ in all the other equations. Therefore, we do not need to consider this equation further in the following.

We now solve the set of equations (16)-(21). We need to rearrange them in a system that can be numerically integrated. There are several equivalent ways to proceed. We could either integrate all the above equations numerically, or we could solve some of them analytically. We choose this second option (in this way, we reduce the number of

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[9] After we eliminated λ, the linearization of the third of (2) coincides with Eq_1.

equations to be numerically integrated, but we generally obtain more involved expressions). We start by solving equations (17) and (18) for $\hat{\chi}$ and $\hat{B}$. First, we need to verify that this can be done. We see that this is the case, since

$$\left| \frac{\partial \{Eq_{01}, Eq_{0i}\}}{\partial \left\{\hat{\chi}, \hat{B}\right\}} \right| = -\frac{\mu^2 \left(1+\mu^2\right) H_a^2}{p_T^2} \neq 0 \tag{22}$$

The explicit solutions are

$$\hat{\chi} = \frac{p_T^2}{2\,\mu^2\,(1+\mu^2)\,H_a^2}\dot{\hat{\Psi}} + \frac{1}{\mu^2\,H_a}\left[p^2 - \frac{\mu^2}{2\,(1+\mu^2)}p_T^2\right]\hat{\Phi} - \frac{p_T^2}{2\,\mu\,(1+\mu^2)\,H_a}\hat{\alpha} - \frac{\hat{\alpha}_0}{\mu}$$

$$\hat{B} = \frac{1}{2\,\mu^2\,H_a^2}\frac{p_T^2}{p_L^2}\left(\frac{p_T^2}{1+\mu^2} - 4\,\mu^2\,H_a^2\right)\dot{\hat{\Psi}} + \frac{1}{\mu^2\,H_a}\frac{p_T^2}{p_L^2}\left[p^2 - \frac{\mu^2}{2\,(1+\mu^2)}p_T^2 - 2\,\left(2+\mu^2\right)\mu^2\,H_a^2\right]\hat{\Phi}$$

$$- \frac{1}{2\,\mu\,H_a}\frac{p_T^2}{p_L^2}\left(\frac{p_T^2}{1+\mu^2} - 4\,\mu^2\,H_a^2\right)\hat{\alpha} - \frac{p_T^2}{p_L^2}\frac{\hat{\alpha}_0}{\mu} \tag{23}$$

Inserting these solutions into the remaining equations (16), (19), (20), and (21), we have now a system of 4 equations in terms of the four unknown modes $\hat{\Psi}$, $\hat{\alpha}$, $\hat{\Phi}$, and $\hat{\alpha}_0$. These equations are explicitly given in appendix A, eqs. (A6) - (A9). As we show in appendix A, by simple algebraic manipulation of these equation we obtain the system

$$\mathcal{M}_\kappa \begin{pmatrix} \ddot{\hat{\alpha}} \\ \dot{\hat{\alpha}}_0 \\ \ddot{\hat{\Psi}} \\ \dot{\hat{\Phi}} \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \end{pmatrix} \quad , \quad \mathcal{M}_\kappa \equiv \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & \kappa_{22} & \kappa_{23} & \kappa_{24} \\ 1 & \kappa_{32} & \kappa_{33} & \kappa_{34} \\ 0 & \kappa_{42} & \kappa_{43} & \kappa_{44} \end{pmatrix} \tag{24}$$

(for reasons explained in the next Section, there are no second time derivatives of $\hat{\alpha}_0$ and $\hat{\Phi}$ in these equations). The entries of $\mathcal{M}_\kappa$, given in eq. (A10) depend on background quantities, while the four expressions at right hand side, given in eq. (A11), are linear combinations of the unknown quantities $\left\{\hat{\alpha}, \dot{\hat{\alpha}}, \hat{\alpha}_0, \hat{\Psi}, \dot{\hat{\Psi}}, \hat{\Phi}\right\}$ (also the coefficients of these linear combinations depend on background quantities). It is now straightforward to invert $\mathcal{M}_\kappa$, and to integrate the system numerically.

It is easy to see that, in general, the solutions of (24) diverge close to horizon crossing. Indeed,

$$\det \mathcal{M}_\kappa = \frac{p_T^2}{p_L^2}\frac{1+\mu^2}{\mu^2}\left[p_L^2 - \left(2+\mu^2\right)H_a\,H_b\right] \tag{25}$$

so that, when we invert this matrix, we encounter a singularity when $p_L = \sqrt{2+\mu^2}\,\sqrt{H_a\,H_b}$.

We evolve the system, starting from the adiabatic vacuum initial conditions deeply inside the horizon (eqs. (A12) and (A16)). The two components of the physical momentum evolve as

$$p_L(t) = \frac{k_L}{a(t)} = p_{L0}\,e^{-H_a\,t}$$

$$p_T(t) = \frac{k_T}{b(t)} = p_{T0}\,e^{-H_b\,t} = p_{T0}\,e^{-\left(1+\mu^2\right)H_a\,t} \tag{26}$$

where p_{L0}, p_{T0} are the values of these components when $t = 0$, and where we have used the background relation (6) between the two Hubble constants. We are free to choose the origin of the time. We set $t = 0$ at the moment in which $\det \mathcal{M}_\kappa = 0$. This fixes $p_{L0} = \sqrt{2+\mu^2}\,\sqrt{H_a\,H_b}$. The mode is then completely specified by giving the ratio p_{T0}/p_{L0}. For definiteness, we choose $\mu = 0.1$ (giving an $O\left(10^{-2}\right)$ anisotropy), and $p_{T0} = H_a$ (so that the two components of the momentum are comparable to each other in the time range considered). We start the numerical evolution at $t = -7/H_a$, so that $H/p = O\left(10^{-3}\right)$ initially, and the modes are well inside the horizon.

The left panel of Figure 1 shows that the modes $\hat{\alpha}_0$ and $\hat{\Phi}$ indeed diverge when $\det \mathcal{M}_\kappa = 0$. The right panel gives the behavior close to the singularity. We give time in log units, so that $t = 0$ is mapped at $-\log\left(-H_a\,t\right) = \infty$. We see that both $\hat{\alpha}_0$ and $\hat{\Phi}$ diverge linearly. The two modes $\hat{\alpha}$ and $\hat{\Psi}$ (not shown here) diverge logarithmically. The modes $\hat{\alpha}$ and $\hat{\alpha}_0$ present an identical divergence in the simplified computation of Section IV. For that simplified analysis, we have also obtained the divergence analytically. It is also clear from there that the solutions diverge for generic initial conditions (we have also verified this in the numerical solutions of the system (24)).
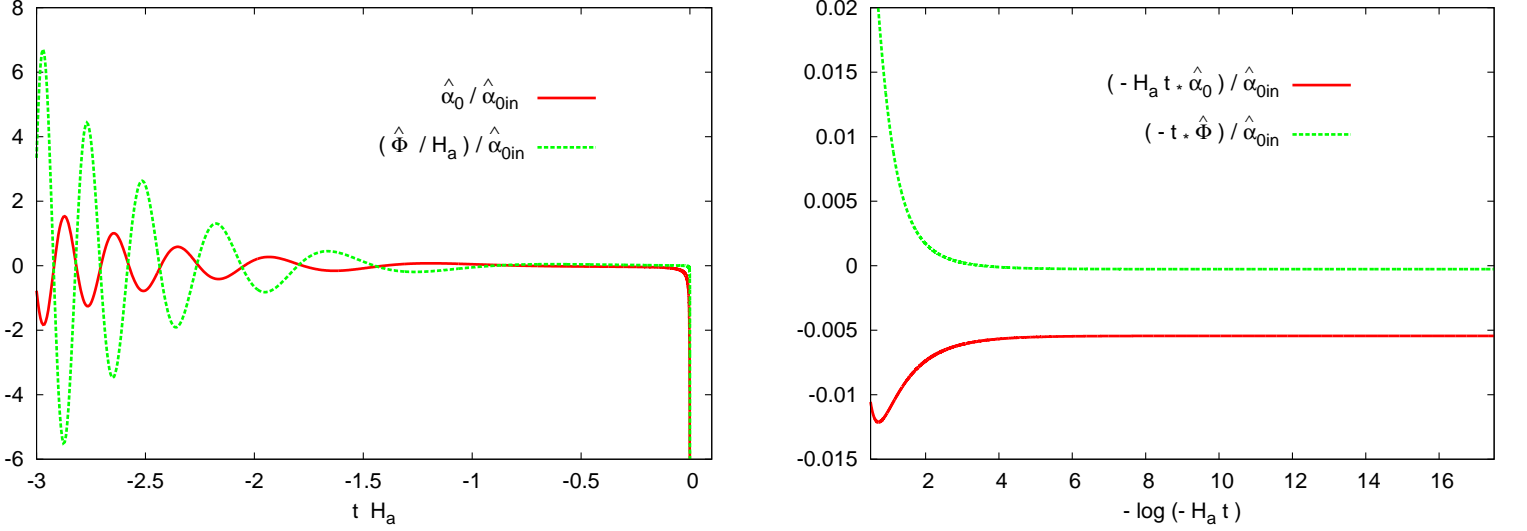
FIG. 1: The right panel shows a zoom of the left panel in the region $-0.6 \leq H_a\,t \leq -10^{-5}$ (the time is shown in log units). Both modes show a $1/t$ divergence.

III. GHOST FROM THE QUADRATIC EQUATION

To gain a better understanding on the nature of the instability, we computed the quadratic action for the perturbations. We inserted the expression (3) and (7) in the action (1). We expanded at the second order in the perturbations. We disregarded the 2d vector modes, which are decoupled from the 2d scalars. We also eliminated the mode $\hat{\alpha}_1$ through the constraint equation (15). We then Fourier transformed the perturbations as in (11). The final result is, up to boundary terms,

$$
\begin{aligned}
S_{2\mathrm{dS}} \;=\; & \frac{1}{2} \int d^3k\, dt\, a\, b^2\, \mathcal{L}_{2\mathrm{dS}} \\[4pt]
\mathcal{L}_{2\mathrm{dS}} \;=\; & \mu^2\, |\dot{\hat{\Psi}}|^2 + \frac{p_T^2}{p_L^2}\, |\dot{\hat{\alpha}}|^2 - H_a \left(\hat{\Psi}^*\, \dot{\hat{\Psi}} + \mathrm{h.c.} \right) - (2+\mu^2)\, H_a \left(\hat{\Phi}^*\, \dot{\hat{\Psi}} + \mathrm{h.c.} \right) + \left(\hat{B}^*\, \dot{\hat{\Psi}} + \mathrm{h.c.} \right) \\[4pt]
& + \mu \left(\hat{\alpha}_0^*\, \dot{\hat{\Psi}} + \mathrm{h.c.} \right) + \frac{p_T^2}{p_L^2}\, H_a \left(\hat{\alpha}^*\, \dot{\hat{\alpha}} + \mathrm{h.c.} \right) + \frac{p_T^2}{p_L^2} \left(\hat{\alpha}_0\, \dot{\hat{\alpha}} + \mathrm{h.c.} \right) - \left[\mu^2\, p_T^2 + (3 + 2\mu^2)\, H_a^2 \right] |\hat{\Psi}|^2 \\[4pt]
& - p_T^2 \left(\hat{\Phi}^*\, \hat{\Psi} + \mathrm{h.c.} \right) + \mu\, p_T^2 \left(\hat{\alpha}^*\, \hat{\Psi} + \mathrm{h.c.} \right) - (2+\mu^2)\, (3+2\mu^2)\, H_a^2\, |\hat{\Phi}|^2 \\[4pt]
& + 2\,(1+\mu^2)\, H_a \left(\hat{\chi}^*\, \hat{\Phi} + \mathrm{h.c.} \right) + (2+\mu^2)\, H_a \left(\hat{B}^*\, \hat{\Phi} + \mathrm{h.c.} \right) + \mu\, H_a \left(\hat{\alpha}_0^*\, \hat{\Phi} + \mathrm{h.c.} \right) \\[4pt]
& + \frac{p_T^2 - 4\mu^2\,(1+\mu^2)\, H_a^2}{2 p_L^2}\, |\hat{\chi}|^2 - \frac{1}{2} \left(\hat{B}^*\, \hat{\chi} + \mathrm{h.c.} \right) - \frac{2\mu\,(1+\mu^2)\, H_a^2}{p_L^2} \left(\hat{\chi}^*\, \hat{\alpha}_0 + \mathrm{h.c.} \right) + \frac{p_L^2}{2 p_T^2}\, |\hat{B}|^2 \\[4pt]
& - \mu\, H_a \left(\hat{B}^*\, \hat{\alpha} + \mathrm{h.c.} \right) - \frac{p_T^2}{p_L^2} \left[p_L^2 - (3+2\mu^2)\, H_a^2 \right] |\hat{\alpha}|^2 + \frac{p_T^2}{p_L^2}\, H_a \left(\hat{\alpha}_0^*\, \hat{\alpha} + \mathrm{h.c.} \right) \\[4pt]
& + \frac{p^2 - 2\,(1+\mu^2)\, H_a^2}{p_L^2}\, |\hat{\alpha}_0|^2
\end{aligned}
\tag{27}
$$

The computation is conceptually straightforward, although technically involved. The resulting action passes several nontrivial checks. Firstly, the perturbations rearrange so that the action can be written solely in terms of gauge invariant combinations. Secondly, extremizing the action with respect to these modes gives precisely the linearized equations (16) - (21). Thirdly, the action correctly identifies the nondynamical variables of the system. The non-dynamical modes are the modes entering in the action (27) without time derivatives. They do not correspond to propagating dynamical degrees of freedom (since the nondynamical modes can be obtained from the dynamical ones by their own equations of motion, that are algebraic - since they enter without time derivatives in the action).

Whether a mode is dynamical or not depends on the initial kinetic terms. For instance, from the field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ we immediately see that the temporal component of the vector field is nondynamical. By direct

inspection, we can also verify that the Ricci scalar does not lead to time derivatives for any of the $g_{0\mu}$ entry of the metric (this is most easily seen through the ADM formalism [32]). The gauge choice of [11] ($E = \Sigma = \tilde{B} = E_i = 0$) preserves all the $\delta g_{0\mu}$ modes, as well as δA_0, namely the modes Φ, χ, B, B_i, α_0. These perturbations are the nondynamical fields in that gauge. Therefore, the combinations $\hat{\Phi}$, $\hat{\chi}$, $\hat{B}$, $\hat{B}_i$, $\hat{\alpha}_0$ given in (9) and (10) are nondynamical (since they are gauge invariant, and since they reduce to the nondynamical modes of the system once the gauge choice of [11] is made). This is explicitly verified by the quadratic action (27). We integrate the nondynamical modes out of the action (27). Namely, we express them as a function of the dynamical modes by using their own algebraic equations of motion. It is instructive to compare this with what happens in the standard case of a scalar field in an isotropic background. In that case, one starts from 11 perturbations (ten in the metric, and one in the scalar field). Four of them are removed by the gauge fixing. Four of them are nondynamical, and are integrated out (for instance, this is what is done in Ref. [30] when the constraint equation (10.53) is used in going from (10.57) to (10.59)). One ends up with three physical propagating modes: the two polarizations of the tensor mode (gravity waves) and the scalar density contrast. In general it is not immediately obvious which combination of gauge invariant perturbations are dynamical (for instance, this is not manifest in the choice of [30]). However, our choice of "hatted" modes makes this manifest.

To be more explicit, the 2d modes entering in (27) can be divided in the two sets of dynamical $Y = \left\{ \hat{\Psi}, \hat{\alpha} \right\}$ and nondynamical $N = \left\{ \hat{\Phi}, \hat{\chi}, \hat{B}, \hat{\alpha}_0 \right\}$ modes. The action (27) is formally of the type

$$S_{YN}^{(2)} = \int dt\, d^3k\, \left[a_{ij}\, \dot{Y}_i^*\, \dot{Y}_j + \left(b_{ij}\, N_i^*\, \dot{Y}_j + \text{h. c.} \right) + c_{ij}\, N_i^*\, N_j + \left(d_{ij}\, \dot{Y}_i^*\, Y_j + \text{h. c.} \right) + e_{ij}\, Y_i^*\, Y_j + \left(f_{ij}\, N_i^*\, Y_j + \text{h. c.} \right) \right] \tag{28}$$

where the matrices formed by the a, c, e coefficients are hermitian. The equations of motion for the nondynamical modes are

$$\frac{\delta S_{YN}^{(2)}}{\delta N_i^*} = 0 \quad \Rightarrow \quad c_{ij}\, N_j = -b_{ij}\, \dot{Y}_j - f_{ij}\, Y_j \tag{29}$$

These equations are precisely Eqs. (16), (17), (18), and (20) of the linearized system that we have solved (notice that, indeed, they not contain any time derivative of the nondynamical modes). We solve them to express the nondynamical modes in terms of the dynamical ones. Inserting these expressions back into (28), we obtain the action for the dynamical modes. This action is formally of the type

$$S_Y^{(2)} = \frac{1}{2} \int dt\, d^3k\, \left[\dot{Y}_i^*\, K_{ij}\, \dot{Y}_j + \left(\dot{Y}_i^*\, X_{ij}\, Y_j + \text{h. c.} \right) - Y_i^*\, \Omega_{ij}^2\, Y_j \right] \tag{30}$$

where (up to boundary terms) the matrices K and Ω^2 are hermitian, while X anti-hermitian. Extremizing this action we find

$$\frac{\delta S_Y^{(2)}}{\delta Y_i^*} = 0 \; \Rightarrow \; K_{ij}\, \ddot{Y}_j + \left(\dot{K}_{ij} + X_{ij} - X_{ji}^* \right) \dot{Y}_j + \left(\dot{X}_{ij} + \Omega_{ij}^2 \right) Y_j = 0$$

$$\Rightarrow \; \ddot{Y}_i = -\left[K^{-1} \left(\dot{K} + X - X^\dagger \right) \right]_{ij} \dot{Y}_j - \left[K^{-1} \left(\dot{X} + \Omega^2 \right) \right]_{ij} Y_j \tag{31}$$

These two equations are precisely the linearized equations (19) and (21), after we have inserted in them the expressions for the nondynamical modes.

It is straightforwards to obtain the expressions of these matrices from the terms in the action (27). However, they are rather involved, and not needed for the present discussion. We list instead the expression for the determinant of the kinetic matrix

$$\det K = \frac{p_T^6\, \left(1 + \mu^2 \right)^2}{2\, \Delta\, p_L^2\, p^2}\, \left[p_L^2 - H_a^2\, \left(1 + \mu^2 \right) \left(2 + \mu^2 \right) \right]$$

$$\Delta \equiv \left[2\, p^2 + \mu^2\, \left(p^2 + p_L^2 \right) \right]^2 - 2 H_a^2\, \left(1 + \mu^2 \right) \left(2 + \mu^2 \right) \left[2 p_L^2\, (1 + \mu^2)^2 + p_T^2\, (2 + \mu^2)\, (1 + 2\mu^2) \right] \tag{32}$$

One can verify that the determinant starts positive at $p \gg H$, and becomes zero precisely when $p_L = \sqrt{2 + \mu^2}\, \sqrt{H_a H_b} = \sqrt{2 + \mu^2}\, \sqrt{1 + \mu^2}\, H_a$. As it is clear from the general form (31) of the equations, one expect that the solutions diverge at this moment. This is precisely what we have found by explicitly solving these equations in the previous Section !

We also see that the determinant of the kinetic matrix becomes negative after this moment. Therefore, one of the physical modes of the system becomes a ghost. This by itself signals an instability of the ACW vacuum.

IV.   SIMPLIFIED COMPUTATION

In this Section we present a simplified stability analysis, which includes only the perturbations of the vector field. This analysis is not rigorous, since the metric perturbations are sourced by these perturbations already at the linearized level. However, as we now show, this simplified study clearly shows the origin of the instability without the involved algebra that is necessary to deal with the perturbations of the metric. We start from the ACW background, given in eqs. (3), (4), and (6). The perturbations of the vector field are as in Section II,

$$\delta A_\mu = (\alpha_0,\, \alpha_1,\, \partial_i \alpha) \tag{33}$$

where we disregard the decoupled 2d vector mode α_i.

The constraint equation enforced by the lagrange multiplier, once linearized in the perturbations, gives $\alpha_1 = 0$ (we recall, that, when the metric perturbations were included, the same constraint enforced $\hat{\alpha}_1 = \mu\,\hat{\Psi}$).

The equations for the vector field (specifically, the second line of (13), in which the lagrange multiplier has been eliminated) then give

$$p_T^2\,\dot{\alpha} - \left[p_L^2 + p_T^2 - 2\left(1+\mu^2\right) H_a^2\right] \alpha_0 = 0$$
$$\ddot{\alpha} + H_a\,\dot{\alpha} - \dot{\alpha}_0 + \left[p_L^2 - 2\left(1+\mu^2\right) H_a^2\right] \alpha - H_a\,\alpha_0 = 0 \tag{34}$$

We proceed by differentiating the first equation with respect to time, and by combining it with the second equation so to eliminate the $\ddot{\alpha}$ term. In this way we get two first order differential equations:

$$\dot{\alpha} = \frac{p_L^2 + p_T^2 - 2\left(1+\mu^2\right) H_a^2}{p_T^2}\,\alpha_0$$
$$\dot{\alpha}_0 = -\frac{\left(1+2\mu^2\right) p_L^2 - 2\left(1+\mu^2\right)\left(3+2\mu^2\right) H_a^2}{p_L^2 - 2\left(1+\mu^2\right) H_a^2}\,H_a\,\alpha_0 - p_T^2\,\alpha \tag{35}$$

This set of equations has a singular point when $p_L^2 = 2\left(1+\mu^2\right) H_a^2 = 2\,H_a\,H_b$. This is the divergence which corresponds to the one occurring at $p_L^2 = \left(2+\mu^2\right) H_a\,H_b$ in the full computation of the two previous Sections. We set the origin of time at the singularity, so that the physical momenta are

$$p_L = \sqrt{2\left(1+\mu^2\right)}\,H_a\,\mathrm{e}^{-H_a\,t} \quad,\quad p_T = p_{T0}\,\mathrm{e}^{-\left(1+\mu^2\right) H_a\,t} \tag{36}$$

Once expanded close to the singularity, the system (35) becomes

$$\dot{\alpha} \approx \alpha_0 \quad,\quad \dot{\alpha}_0 \approx -\frac{\alpha_0}{t} - p_{T0}^2\,\alpha \tag{37}$$

which we can combine into (we obtain the same result if we first combine the two equations (35) into a unique equations for α, and we then expand that one)

$$\ddot{\alpha} + \frac{\dot{\alpha}}{t} + p_{T*}^2\,\alpha \approx 0 \tag{38}$$

which is in turns solved be

$$\alpha \approx C_1\,J_0\left(p_{T*}\,t\right) + C_2\,Y_0\left(p_{T*}\,t\right) \tag{39}$$

where $C_{1,2}$ are constants to be determined from initial conditions. While the J_0 solution is constant at $t = 0$, the Y_0 solution has a logarithmic divergence. In principle, one may imagine arranging the initial conditions, so that the solution will be regular ($C_2 = 0$) at $t = 0$. One would need to do so for both the real and imaginary parts of α and for all modes (namely, for all values of the comoving momenta). There is however no physical reason for this tuning; the initial conditions are given when the mode is deeply inside the horizon, well before the equations become singular (there is no reason why the mode should initially "know" about the singularity that will happen close to horizon crossing). Moreover, as we will see, the solution is clearly divergent ($C_2 \neq 0$) if the initial conditions are chosen in the adiabatic vacuum. Therefore, α has a logarithmic divergence, and $\alpha_0 \approx \dot{\alpha}$ has a linear $1/t$ divergence close to horizon crossing, which indicates that the background solution is unstable. This result is confirmed by the numerical solutions below. Moreover, this degree of divergence is the same as the one obtained in the complete computation of Subsection (II C).

To find the initial conditions, and to understand the reason for the instability, we compute the quadratic action for the perturbations. We find (up to boundary terms)

$$S_s = \frac{1}{2} \int d^3k \, dt \, a \, b^2 \left\{ p_T^2 \, |\dot{\alpha}|^2 - p_T^2 \, (\alpha_0^* \dot{\alpha} + \text{h.c.}) - p_T^2 \left[p_L^2 - 2 \left(1 + \mu^2 \right) H_a^2 \right] |\alpha|^2 \right.$$

$$\left. + \left[p_L^2 + p_T^2 - 2 \left(1 + \mu^2 \right) H_a^2 \right] |\alpha_0|^2 \right\} \tag{40}$$

This actions leads to the equations (35), when it is extremized. The mode α_0 appears without any time derivatives in the action, therefore it can be integrated out by solving its equation of motion, which gives

$$\alpha_0 = \frac{p_T^2}{p_L^2 + p_T^2 - 2 \left(1 + \mu^2 \right) H_a^2} \, \dot{\alpha} \tag{41}$$

Inserting this solution back into the action we get

$$S_s = \frac{1}{2} \int d^3k \, dt \, a \, b^2 \, p_T^2 \, \left(p_L^2 - 2 \left(1 + \mu^2 \right) H_a^2 \right) \left[\frac{|\dot{\alpha}|^2}{p_L^2 + p_T^2 - 2 \left(1 + \mu^2 \right) H_a^2} - |\alpha|^2 \right] \tag{42}$$

In the early-time/UV limit when $p_{L,T}^2 \gg H_a^2$, this action is ghost free and α is stable. However, there is a moment later in the evolution, close to horizon crossing, when the longitudinal physical momentum becomes $p_L = \sqrt{2 \left(1 + \mu^2 \right)} \, H_a$ and the action vanishes. This generates the singularity in the equations (35). In addition, α becomes a ghost after the action vanishes, since the kinetic term becomes negative. [10]

The canonical variable of the system is

$$\alpha_c \equiv \sqrt{a} \, b \, p_T \sqrt{\frac{p_L^2 - 2 \left(1 + \mu^2 \right) H_a^2}{p_L^2 + p_T^2 - 2 \left(1 + \mu^2 \right) H_a^2}} \, \alpha \tag{43}$$

Inserting this into the action (42) we get (up to a boundary term)

$$S_s = \frac{1}{2} \int d^3k \, dt \, \left(|\dot{\alpha}_c|^2 - \omega_c^2 \, |\alpha_c|^2 \right) \tag{44}$$

The exact expression of ω_c is rather involved, and we do not report it here. The first two terms in the early time expansion ($H \ll p$) are

$$\omega_c = p \left[1 - \frac{\left(9 + 8\mu^2 \right) p_L^4 + \left(3 + 2\mu^2 \right)^2 p_T^4 + 2 \left(9 + 10\mu^2 - 4\mu^4 \right) p_L^2 \, p_T^2}{8p^4} \, \frac{H_a^2}{p^2} + \text{O}\left(\frac{H_a^4}{p^4} \right) \right] \tag{45}$$

The frequency ω_c is adiabatically evolving at early times, $\dot{\omega}_c \ll \omega_c = \text{O}\left(H/p \right) \ll 1$. Therefore, we can set the initial conditions for the canonical variable α_c in the adiabatic vacuum

$$\alpha_{c, \text{in}} = \frac{1}{\sqrt{2 \, \omega_c}} \, e^{-i \int^t \omega_c \, dt} \tag{46}$$

This gives the initial values of α_c and $\dot{\alpha}_c$. From (44) and its time derivative we then obtain the initial values of α and $\dot{\alpha}$. Finally, from (41) we obtain the initial value of α_0. Specifically, we find

$$\text{Im} \left(\alpha_0 / \alpha \right)_{\text{in}} = -\frac{p_T^2}{p} \left[1 + \text{O}\left(\frac{H^2}{p^2} \right) \right] \quad , \quad \text{Re} \left(\alpha_0 / \alpha \right)_{\text{in}} = -\frac{\mu^2 \, p_T^4}{2 \, p^3} \, \frac{H_a}{p} \left[1 + \text{O}\left(\frac{H^2}{p^2} \right) \right] \tag{47}$$

The overall phase of the modes is unphysical, and we can always choose α_{in} to be real. Since the coefficients of (35) are real, we need to evolve this system twice, one for the real part and one for the imaginary parts of the modes. We did so starting from $t = -7/H_a$, so that $H/p = \text{O}\left(10^{-3} \right)$ initially. For definiteness, we take $\mu = 0.1$, $p_{T0} = H_a$ (as in the full computation leading to Figure 1). We show the solutions for the real part in Figure 2. The right panel is a close up near the singularity (which, in logarithmic units, occurs at $-\ln \left(-H_a \, t \right) = +\infty$). The two solutions have been re-scaled so to make the degree of divergence manifest. The imaginary part presents an analogous divergence.

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[10] The kinetic term also diverges when $p = \sqrt{2 \left(1 + \mu^2 \right)} \, H_a$ (where $p = \sqrt{p_L^2 + p_T^2}$); this is due to the fact that α_0 cannot be integrated at this point. However, this happens after the singularity we are interested in.
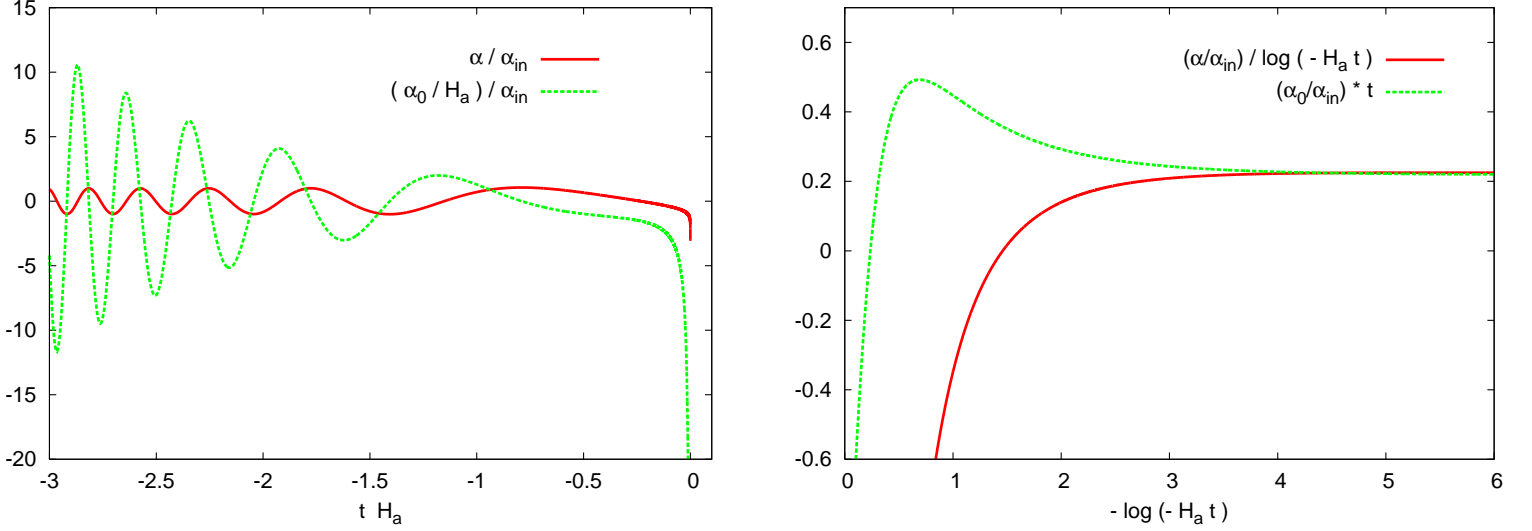
FIG. 2: The right panel shows a zoom of the left panel in the region $-1 \leq H_a t \leq -2.5 \cdot 10^{-3}$ (the time is shown in log units). Both modes blow up when the system (35), becomes singular, $\alpha \propto \log t$ and $\alpha_0 \propto 1/t$, in agreement with the analytic approximate solution (39). Notice also that $\alpha_0 \approx \dot{\alpha}$ close to the singularity, in agreement with the analytic computation.

V.   APPEARANCE OF A GHOST IN OTHER THEORIES

We have performed a stability analysis also of the models [18, 26, 27, 28]. The computations for these models are more involved than the one presented here, since they involve a larger number of perturbations. [11] They will be reported elsewhere. We anticipate that these models have a ghost close to horizon crossing. In addition, for the two models [27, 28] the linearized equations blow up close to horizon crossing, analogously to what happens in the ACW case.

The three models [26, 27, 28] are characterized by a nonminimal coupling of one or more vector fields to the curvature. The presence of the ghost is not due to an anisotropic geometry (in fact, the background evolution can be taken isotropic in [27], by having $3N$ mutually orthogonal vectors, and it is isotropic in [26]) but rather to the sign of the coupling to the curvature. The action for the vector field(s) is

$$S = \int d^4x \sqrt{-g} \left[-\frac{1}{4}F^2 + \frac{1}{12} R A^2 \right] \tag{48}$$

where the $1/12$ coefficient allows for a slow roll of the vector in [27, 28] (see the discussion in [29]) and the production of a (nearly) scale invariant spectrum of perturbations in [26]. [12] For simplicity, we assume that the vector field has no vev, and the background is pure de Sitter. Using conformal time η,

$$S = \int d\eta \, d^3x \left[-\frac{1}{4}F^2 - \frac{M^2}{2} A^2 \right] \tag{49}$$

where $M^2 = -2 a^2 H^2$, and where the indices are risen with the inverse of the Minkowski metric $\eta_{\mu\nu} = \text{diag}(-1, +1, +1, +1)$.

The fact that the longitudinal vector (48) is a ghost in this model depends on the sign of the mass term. At first, it is not immediate why a mass term, and not a kinetic term, controls the nature of a mode. The main reason is that the longitudinal degree of freedom is present only for $M^2 \neq 0$, since, for vanishing mass, the action (48) is U(1) invariant, and the vector has only the two transverse degrees of freedom. We present three arguments to show that $M^2 < 0$ leads to a ghost. [13] In all the discussion below, $M \equiv |M^2| > 0$. Moreover, only for this Section, $k^2 \equiv k^\mu k_\mu$, $\vec{k}^2 \equiv k_i k_i$.

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[11] A complete computation is presumably even more complicated for models where the kinetic term for the vector field is highly noncanonical, see for instance Ref. [33].

[12] See [34] for a similar study, based on a different model of vector fields.

[13] We thank A. Vainshtein for suggesting us to use the first two arguments, and for discussions. We thank A. J. Tolley for suggesting us to use the second argument. The third argument was presented in our previous work [29].

The most standard way to compute how many states there are in a theory, and what their nature is, is by computing the propagator. The quadratic lagrangian in (49) can be cast in the form $(1/2)A^\mu\, P^{-1}_{\mu\nu}\, A^\nu$, where

$$P_{\mu\nu} = -\frac{\eta_{\mu\nu} + \frac{k_\mu k_\nu}{M^2}}{k^2 + M^2} \tag{50}$$

is the propagator. In general, the propagator needs to be diagonalized. However, we can choose a frame in which it is diagonal (clearly, the number and nature of physical modes does not depend on the frame). For a positive M^2, the pole is at $k^2 = -M^2 < 0$, and we can go in the rest frame, where $k^\mu = -k_\mu = (M, 0, 0, 0)$. In this case, $-\left(\eta_{\mu\nu} + k_\mu k_\nu/M^2\right) = \mathrm{diag}\,(0, -1, -1, -1)$, indicating that the theory has three well behaved physical particles (-1 indicates a well behaved mode due to the signature; cf. the propagator for a scalar particle). For $M^2 < 0$, we cannot go in the rest frame; however, we can choose a frame where the energy vanishes, $k^\mu = k_\mu = (0, 0, 0, M)$. In this case, $-\left(\eta_{\mu\nu} + k_\mu k_\nu/M^2\right) = \mathrm{diag}\,(1, -1, -1, 0)$, indicating that one mode (the longitudinal vector) is a ghost.

The ghost also appears in the Stuckelberg formalism. For simplicity, here M is treated has a constant (the time dependence of M does not modify the quadratic kinetic term, but it complicates the diagonalization). If we redefine

$$A_\mu = B_\mu + \frac{1}{M}\partial_\mu\phi \tag{51}$$

we promote the action (49) to a gauge invariant action, with the symmetry

$$\phi \to \phi + \xi \quad , \quad B_\mu \to B_\mu - \frac{1}{M}\partial_\mu\phi \tag{52}$$

The action (49) is recovered in the unitary gauge, $\phi = 0$. But one can also choose a gauge in which B_μ is transverse, $\partial^\mu B_\mu = 0$. In this gauge, the action of the system is

$$S = \int d\eta\, d^3x \left[-\frac{1}{4}F\,(B)^2 - \frac{M^2}{2}B^2 \mp \frac{1}{2}\,(\partial\phi)^2 \right] \tag{53}$$

where the field strength $F_{\mu\nu}$ does not contain ϕ, and where the kinetic term of ϕ has opposite sign to M^2. We stress that the two actions (49) and (53) describe the same theory in two different gauges. We again see that the longitudinal component ϕ is a normal field for $M^2 > 0$, and a ghost for $M^2 < 0$.

A third way to find the ghost is to decompose the vector as $A_\mu = (\alpha_0,\, \partial_i\alpha + \alpha_i)$, where $\partial_i\alpha_i = 0$. The action (49) then separates in two decoupled pieces,

$$\begin{aligned}
S &= \int d\eta d^3x \left\{ \frac{1}{2}\left[\alpha_i'^2 - (\partial_i\alpha_j)^2 - M^2\alpha_i^2 \right] + \frac{1}{2}\left[(\partial_i\alpha')^2 - 2\partial_i\alpha'\partial_i\alpha_0 - M^2\,(\partial_i\alpha)^2 + (\partial_i\alpha_0)^2 + M^2\,\alpha_0 \right] \right\} \\
&= \int d\eta d^3k \left\{ \frac{1}{2}\left[|\alpha_i'|^2 - \left(\vec{k}^2 + M^2\right)|\alpha_i|^2 \right] + \frac{1}{2}\left[\vec{k}^2|\alpha'|^2 - \vec{k}^2\,(\alpha'^*\,\alpha_0 + \mathrm{h.c.}) - M^2\,\vec{k}^2|\alpha|^2 + \left(\vec{k}^2 + M^2\right)|\alpha_0|^2 \right] \right\}
\end{aligned} \tag{54}$$

where prime denotes derivative wrt conformal time, and in the second line we have Fourier transformed the modes as in (11). The action splits in two decoupled parts. The first part governs the two transverse polarizations, which are well behaved. The second part contains only one physical mode, since α_0 enters without time derivatives and must be integrated out. The equation of motion for this field obtained from (54) is $\alpha_0 = \left[\vec{k}^2/\left(\vec{k}^2 + M^2\right)\right]\alpha'$. Inserting this back into the second part of (54), we obtain

$$S_{\mathrm{longitudinal}} = \int d\eta\, d^3k\, \frac{\vec{k}^2\, M^2}{2}\left[\frac{|\alpha'|^2}{\vec{k}^2 + M^2} - |\alpha|^2 \right] . \tag{55}$$

Again, we see that the longitudinal vector is well behaved if $M^2 > 0$, and a ghost if $M^2 < 0$. As a check that this action is correct, we can compute the equations of motion enforced by it,

$$\alpha'' + \frac{2k^2 aH}{k^2 + M^2}\alpha' + \left(k^2 + M^2\right)\alpha = 0 \tag{56}$$

where we have used the fact that, for $M^2 = -2a^2H^2$, one has $\left(M^2\right)' = 2aHM^2$. This is precisely the equation of motion for the longitudinal vector already given in the literature, see for instance eq. (45) of [26] (notice that, in their convention, R has the opposite sign wrt our convention).

VI.   DISCUSSION

We have shown by explicit computations that the ACW background solution is unstable. We did so in Section II by writing down and solving the linearized equations for the most general set of perturbations of the model. These equations, and their solutions, diverge at some point t_* close to horizon crossing (this occurs for any mode in the Fourier decomposition; we stress that all the modes are decoupled from each other at the linearized level). The computations of Section II prove by themselves that the background is unstable (it is possible that the divergency does not occur at the full nonlinear level. However, when the nonlinear interactions become important, the solution will be substantially different from the ACW background solution). The computations the we have presented next clarify the origin of the instability. Specifically, we have shown in Section III that one dynamical variable of the system becomes a ghost at t_*. The kinetic term for this mode vanishes at t_*, which explains why the corresponding equation of motion, and the solutions, diverge at this point. We stress that the presence of a ghost indicates on its own an instability of the vacuum (see below). Our analysis is conceptually straightforward, but algebraically very involved. For this reason, we have also shown that our expressions pass several nontrivial checks. To gain a further understanding on the instability, we have also given, in Section IV, a simplified stability study, including only the perturbations of the vector field. The algebra involved is now considerably simpler than in the previous case. We verified that this system presents the same instability of the complete computation: the solutions diverge when one mode becomes a ghost. As we mentioned in the Introduction, the metric perturbations complicate the computation, but do not remove the intrinsic instability of the model, which is due to the vector field.

In a loose sense, the ACW model is "doubly unstable", since the linearized equations for the perturbations diverge - we call this "instability I" - and since it has a ghost - "instability II" (in a generic model, either instability may appear without the other one). Motivated by this finding, we investigated whether other models with vector fields playing some role during inflation are also unstable. To our knowledge, the fist model where anisotropic inflation is due to a vector is the one of Ford [18]. The original proposal studied a subset of perturbations in the infinite wavelength limit. A general computation, performed along the line of the one presented here, shows that this model contains a ghost in the short wavelength regime (it has the instability II mentioned above, but not the instability I). The models [26, 27, 28] are characterized by a nonminimal coupling of the vector to the curvature, with a precise factor. We have showed in Secton V why this leads to a ghost (the presence of a ghost is confirmed by the complete computation of the perturbations; we stress that the instability is due to the specific sign mass term, and not to the anisotropic expansion). The models [27, 28] present also the instability of the type I, as defined above, although this is not manifest. The detailed computations will be presented elsewhere.

An instability of the type I is a clear problem of the linearized theory. The implications of having a ghost instead require an understanding of the full theory. Such a study is beyond the goal of the present work. Nonetheless, we would like to provide some general comments. Although, contrary to what we have presented so far, these final considerations are not based on explicit computations, we believe that they are worth mention, and, possibly, more specific future investigations.

The most obvious problem associated to a ghost is the instability of the vacuum. If a ghost is coupled to a normal field (and, in all the above theories, there are at least gravitational couplings), the vacuum will decay in ghost-nonghost excitations, with a rate which is UV divergent. This is simply due to phase space considerations. In normal situations, a source of any given energy can produce quanta up to some given momentum; this cuts-off the phase space available to the decay products. However, the quanta of a ghost field have negative energy: the higher their momentum, the more negative the energy is. Therefore, even a zero energy source decays into ghost-nonghost excitations conserving both energy and momentum. One may hope that the nonlinear interactions may somewhat arrange to cancel the total decay rate. However, this does not appear likely. The usual approach is to regard theories with ghosts as effective theories, valid only below some energy scale. Inflationary predictions heavily rely on the initial conditions; for instance, a slowly rolling inflaton field results in a nearly scale invariant spectrum of cosmological perturbations because the initial modes of the inflaton have amplitude $|\delta\phi| \propto k^{-1/2}$. This is due to the choice of an initial adiabatic vacuum in the quantized theory for the perturbations, which is performed in the deep UV regime (energy $\gg H^{-1}$) [30]. In presence of a cut-off, the initial adiabatic vacuum cannot be imposed at arbitrarily early times, and, depending on the cut-off, it may not be possible to impose it at all. This casts doubts on the phenomenological predictions obtained for such models.

In fact, all these theories require a cut-off which makes them invalid at high energies, irrespectively of the sign of the mass term. [14] We can see this based on the behavior of massive vector fields at high energies. The models

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[14] Several of the considerations presented here emerged from discussions with N. Kaloper, who we thank for either pointing them out, or for stressing their relevance. We also acknowledge very useful discussions with D.H. Lyth on these issues.

studied here have a gauge invariance that is broken in a hard way by the explicit mass term $M^2 A^2$ for the vector. It is well known that, in such cases, the interactions of the longitudinal bosons violate unitarity at a scale which is parametrically set by M, leading to a quantum theory out of control. For the present models, M is the Hubble rate or below, so that the entire sub-horizon regime may be ill-defined. Although we are aware of explicit computations of this problem only in Minkowski spacetime, we believe that it applies also to the inflationary case, if one has unbroken Lorentz invariance and transitions to a locally flat frame (moreover, during inflation the mass term for the vector is nearly constant, and the momentum is adiabatically varying in the sub horizon regime); this issue is less transparent for the specific ACW model, in which the constraint enforced by λ will modify the interactions of the vector; however, we believe that this concern is legitimate also for that model. One may then adopt the approach of ignoring the ill behaved longitudinal vector mode, and concentrate on the other degrees of freedom in the model. We do not believe that this approach is justified. At high energies, the longitudinal mode will also interact strongly with the other fields in the theory (this will renormalize the coupling constants). Then, depending on exactly when this happens, the quantum theory of the perturbations may be out of control throughout the entire short wavelength regime. If this is the case, all initial conditions would become unjustified, and the theory would lose its predictive power.

We are well aware that these problems, if indeed they are present, will affect also the finding of the present work. However, in our opinion, the computations presented above would not lose their importance. One reason for this is that the instabilities that we have found have not emerged so far in the literature. Different studies have adopted different approximation schemes, which are very useful in the case of scalar fields, as for instance studying separately the short or long wavelength regimes [23], approximating the source with fluids [35] [15], or using the ΔN formalism [26] (which allows to follow the evolution of the perturbations only in the long wavelength regime). Although such studies are perfectly legitimate in the regime of their validity, and can also give very useful results in the present case, one should be aware that these models require extra checks. We believe that our work offers some nontrivial technical advance in this respect.

A more important reason is that "curing" a theory which has a hard vector mass and a ghost is more problematic than curing a theory with only a hard vector mass. The most immediate UV completion of a theory with a hard mass is through a higgs mechanism. The mass would be then due to the vev of a scalar field that becomes dynamical above the scale M. In this way the theory remains under control also in the short wavelength regime, and one can apply all the standard computations valid for scalar fields during inflation. However, if M^2 needs to have the wrong sign, the scalar field in this UV completed theory needs to be a ghost. UV completing a theory with a condensed ghosts may present analogous problems to the UV completion [36] of the ghost condensation model of [37]. We hope to return to this issue in some future investigation.

APPENDIX A: EXPLICIT EXPRESSIONS FOR THE COMPUTATION OF SUBSECTION II C

We write here some explicit steps in the linearized computation of the perturbations performed in Subsection II C.

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[15] Clearly, the microscopic description of the fluids considered in [35] does not need to involve vector fields, although this has been considered as one of the possibilities in that work.

The explicit form of equations (14) is

$$\text{Eq}_{00}: \quad -\frac{1}{M_p}\left\{\left(2+\mu^2\right)H_a\,\dot{\hat{\Psi}}+p_T^2\,\hat{\Psi}+\left(2+\mu^2\right)\left(3+2\mu^2\right)H_a^2\,\hat{\Phi}-2\left(1+\mu^2\right)H_a\,\hat{\chi}\right.$$

$$\left.-\left(2+\mu^2\right)H_a\,\hat{B}-\mu\,H_a\,\hat{\alpha}_0-\mu\,H_a\left(\dot{\hat{\alpha}}_1-\mu\,\dot{\hat{\Psi}}\right)-\mu\,H_a^2\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)\right\}=0$$

$$\text{Eq}_{01}: \quad -\frac{i\,p_L\,a}{M_p}\left\{2\left(1+\mu^2\right)H_a\,\hat{\Phi}+\frac{1}{2p_L^2}\left[p_T^2-4\mu^2\left(1+\mu^2\right)H_a^2\right]\hat{\chi}-\frac{\hat{B}}{2}-2\mu\left(1+\mu^2\right)\frac{H_a^2}{p_L^2}\,\hat{\alpha}_0\right\}=0$$

$$\text{Eq}_{0i}: \quad -\frac{i\,p_{Ti}\,b}{M_p}\left[\dot{\hat{\Psi}}+\left(2+\mu^2\right)H_a\,\hat{\Phi}-\frac{\hat{\chi}}{2}+\frac{p_L^2}{2p_T^2}\,\hat{B}-\mu\,H_a\,\hat{\alpha}+\mu\,H_a\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)\right]$$

$$+\frac{b}{2}\left[-p_L^2\,\dot{\hat{B}}_i+H_a\,p_L^2\left(1+\mu^2\right)\hat{B}_i+p^2\,\hat{B}_i-2\,i\,\mu\,\frac{H_a}{b}\,p_L\,\hat{\alpha}_i\right]=0$$

$$\text{Eq}_{11}: \quad -\frac{\mu\,a^2}{M_p}\left\{\ddot{\hat{\alpha}}_1+2\left(2+\mu^2\right)H_a\,\dot{\hat{\alpha}}_1+p_T^2\,\hat{\alpha}_1+\dot{\hat{\alpha}}_0+\left(3+2\mu^2\right)H_a\,\hat{\alpha}_0-p_T^2\,\hat{\alpha}-\mu\,H_a\,\dot{\hat{\Psi}}\right.$$

$$-\frac{2+\mu^2}{\mu}\,H_a\,\dot{\hat{\Phi}}+\frac{1}{\mu}\left[p_T^2-\left(2+\mu^2\right)\left(3+2\mu^2\right)H_a^2\right]\hat{\Phi}+\frac{\dot{\hat{B}}}{\mu}+\frac{3+2\mu^2}{\mu}\,H_a\,\hat{B}$$

$$\left.+\left(9+8\mu^2\right)H_a^2\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)\right\}=0$$

$$\text{Eq}_{1i}: \quad \frac{a\,b\,p_{Ti}}{2\,p_L\,M_p}\left[\dot{\hat{\chi}}+3H_a\,\hat{\chi}+\frac{p_L^2}{p_T^2}\left[\dot{\hat{B}}+\left(3+4\mu^2\right)H_a\,\hat{B}\right]-2\,\mu\,H_a\left(\dot{\hat{\alpha}}+3H_a\,\hat{\alpha}\right)+2p_L^2\,\hat{\Phi}\right.$$

$$\left.-4\,\mu^3\,H_a^2\,\hat{\alpha}-2\,\mu\,H_a\,\hat{\alpha}_0\right]$$

$$-\frac{i\,a\,b\,p_L}{2}\left[\ddot{\hat{B}}_i+\left(1+2\mu^2\right)H_a\,\dot{\hat{B}}_i-\dot{\hat{B}}_i-\left(2+3\mu^2\right)H_a\,\hat{B}_i\right.$$

$$\left.+\left[p_T^2-H_a^2\left(1+\mu^2\right)\left(2+3\mu^2\right)\right]\hat{B}_i+2\,i\,\frac{H_a\,\mu}{b\,p_L}\left(\dot{\hat{\alpha}}_i+2\,H_a\left(1+\mu^2\right)\hat{\alpha}_i\right)\right]=0$$

$$\text{Eq}_{ij}: \quad \frac{b^2}{M_p}\left\{\left[\ddot{\hat{\Psi}}+\left(3+2\mu^2\right)H_a\,\dot{\hat{\Psi}}+p_T^2\,\hat{\Psi}+\left(2+\mu^2\right)H_a\,\dot{\hat{\Phi}}-\left[p^2-H_a^2\left(2+\mu^2\right)\left(3+2\mu^2\right)\right]\hat{\Phi}-\dot{\hat{\chi}}\right.\right.$$

$$\left.-\dot{\hat{B}}-\left(3+2\,\mu^2\right)H_a\,\hat{B}-\left(3+\mu^2\right)H_a\,\hat{\chi}+\mu\,H_a\,\dot{\hat{\alpha}}_1+\mu\,H_a\,\hat{\alpha}_0+\mu\,H_a^2\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)\right]\delta_{ij}$$

$$\left.-p_{Ti}\,p_{Tj}\left[\hat{\Psi}-\hat{\Phi}-\frac{1}{p_T^2}\left(\dot{\hat{B}}+\left(3+2\mu^2\right)H_a\,\hat{B}\right)\right]\right\}$$

$$+i\,b^2\left[p_{(Ti}\,\dot{\hat{B}}_{j)}+\left(2+\mu^2\right)H_a\,p_{(Ti}\,\hat{B}_{j)}+p_L^2\,p_{(Ti}\,\hat{B}_{j)}\right]=0$$

$$\text{Eq}_0: \quad -\frac{i\,p_T^2}{p_L}\left\{\dot{\hat{\alpha}}+\frac{p_L^2}{p_T^2}\,\hat{\alpha}_1+H_a\,\hat{\alpha}+\frac{1}{p_T^2}\left[p^2-2\left(1+\mu^2\right)H_a^2\right]\hat{\alpha}_0+\mu\,\frac{p_L^2}{p_T^2}\,H_a\,\hat{\Phi}\right.$$

$$\left.-2\,\mu\left(1+\mu^2\right)\frac{H_a^2}{p_T^2}\,\hat{\chi}+\frac{p_L^2}{p_T^2}\,H_a\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)\right\}=0$$

$$\text{Eq}_1: \quad \frac{4\left(1+\mu^2\right)H_a^2}{a}\left(\hat{\alpha}_1-\mu\,\hat{\Psi}\right)=0$$

$$\text{Eq}_i: \quad =-\frac{p_{Ti}}{b\,p_L}\left[\ddot{\hat{\alpha}}+3\,H_a\,\dot{\hat{\alpha}}+\dot{\hat{\alpha}}_0+\left(p_L^2-2\mu^2\,H_a^2\right)\hat{\alpha}-p_L^2\,\hat{\alpha}_1+\mu\,\frac{p_L^2}{p_T^2}\,H_a\,\hat{B}+2\,H_a\,\hat{\alpha}_0\right]$$

$$+\frac{i\,M_p}{b}\left[p_L\,\mu\,H_a\left(\dot{\hat{B}}_i-\left(1+\mu^2\right)H_a\,\hat{B}_i-\hat{B}_i\right)-\frac{i}{b}\left[\ddot{\hat{\alpha}}_i+H_a\,\dot{\hat{\alpha}}_i+\left(p^2-2\,H_a^2\left(1+\mu^2\right)\right)\hat{\alpha}_i\right]\right]=0$$

$$(A1)$$

where the notation on the momenta is explained after eq.(11), and where $x_{(i}\,y_{j)} \equiv \frac{1}{2}\left(x_i\,y_j + x_j\,y_i\right)$ denotes symmetrization. Notice that it has been possible to write all the above equations in terms of gauge invariant combinations. This is a nontrivial check on our algebra.

We recall that the indices $i, j = 1, 2$ span the symmetric $y - z$ space. Equations carrying these indices contain both 2d scalar and 2d vector modes. However, it is easy to see that each of these equations separates in two independent equations, one for the 2d scalars and one for the 2d vectors: the equations in (A1) carrying an i index have the following structure

$$\mathrm{Eq}_{0i} = k_{Ti}\,\mathcal{S}_1 + \mathcal{V}_{1i}\,, \quad \mathrm{Eq}_{1i} = k_{Ti}\,\mathcal{S}_2 + \mathcal{V}_{2i}\,, \quad \mathrm{Eq}_{ij} = \delta_{ij}\,\mathcal{S}_3 + k_{Ti}\,k_{Tj}\,\mathcal{S}_4 + k_{(Ti}\,\mathcal{V}_{3j)}\,, \quad \mathrm{Eq}_i = k_{Ti}\,\mathcal{S}_5 + \mathcal{V}_{4i} \quad (\text{A2})$$

where the expressions $\mathcal{S}_{1,\ldots,5}$ contain the 2d scalar, and the expressions $\mathcal{V}_{1,\ldots,4}$ contain the 2d vector modes. Due to the transversality conditions of the 2d vectors,

$$k_{Ti}\,\mathcal{V}_{1i} = \cdots = k_{Ti}\,\mathcal{V}_{4i} = 0 \quad \Rightarrow \quad \mathcal{V}_{1i} = \cdots = \mathcal{V}_{4i} = 0 \quad (\text{A3})$$

so that we see explicitly that the equations for the 2d scalars and the 2d vectors are indeed decoupled from each other. We have verified that the system of 2d vectors does not contain instabilities. For brevity, we disregard these modes in the following, and we concentrate only on the 2d scalars.

We note that not all the equations in the system (A1) are independent. Indeed, the Bianchi identities can be linearized to give

$$\nabla_\mu \left[G^\mu_\nu - \frac{1}{M_p^2}\,T^\mu_\nu \right] = 0 \quad \rightarrow \quad g^{(0)\,\mu\alpha}\,\nabla^{(0)}_\mu\,\mathrm{Eq}_{\alpha\nu} = 0 \quad (\text{A4})$$

where $\nabla^{(0)}$ denotes the covariant derivative constructed from the background metric $g_{\mu\nu}$. These can be written as explicit relations between the linearized equations (14)

$$\nu = 0: \quad \frac{d}{dt}\,\mathrm{Eq}_{00} + \left(3 + 2\mu^2\right)H_a\,\mathrm{Eq}_{00} + i\left(\frac{p_L}{a}\,\mathrm{Eq}_{01} + \frac{p_{Ti}}{b}\,\mathrm{Eq}_{0i}\right) + \frac{H_a}{a^2}\,\mathrm{Eq}_{11} + \frac{\left(1 + \mu^2\right)H_a}{b^2}\,\delta^{ij}\,\mathrm{Eq}_{ij} = 0$$

$$\nu = 1: \quad \frac{d}{dt}\,\mathrm{Eq}_{01} + \left(3 + 2\mu^2\right)H_a\,\mathrm{Eq}_{01} + i\left(\frac{p_L}{a}\,\mathrm{Eq}_{11} + \frac{p_{Ti}}{b}\,\mathrm{Eq}_{1i}\right) = 0$$

$$\nu = i: \quad \frac{d}{dt}\,\mathrm{Eq}_{0i} + \left(3 + 2\mu^2\right)H_a\,\mathrm{Eq}_{0i} + i\left(\frac{p_L}{a}\,\mathrm{Eq}_{1i} + \frac{p_{Tk}}{b}\,\mathrm{Eq}_{ki}\right) = 0 \quad (\text{A5})$$

We explicitly checked that the system of equations (14) satisfies the above identities (which is a further nontrivial check on our algebra). Therefore, (A5) verifies that some of the linearized equations are redundant. By inserting the decomposition (A2) into these identities (disregarding the 2d vector parts of the equations), we see that the $i = 2, 3$ identities become equal to each other. We also see that we can use the three nontrivial identities ($\nu = 0, 1, 2$) to express $\mathcal{S}_2,$, $\mathcal{S}_3$ and $\mathcal{S}_4$ in terms of the other equations. Therefore, the Eq_{1i} and Eq_{ij} equations in (A1) can be obtained from the other ones, and can be disregarded (obviously, one could equivalently choose to disregard some other equations, as long as it is possible to express them in terms of the remaining ones in (A5)).

This leads us to eqs. (15)-(21) of the main text. We solve equations (17) and (18) for $\hat{\chi}$ and $\hat{B}$. The other equations become

$$\frac{1}{H_a}\left[-\frac{1}{\mu^2}\,\frac{p_T^2\,p^2}{p_L^2} + \frac{p_T^4}{2\left(1 + \mu^2\right)p_L^2} + \left(2 + \mu^2\right)\frac{p_L^2 + 2\,p_T^2}{p_L^2}H_a^2 \right]\dot{\hat{\Psi}} + p_T^2\,\hat{\Psi}$$

$$+ \left[-\frac{2\left(1 + \mu^2\right)}{\mu^2}\,p^2 - \frac{2\,p_T^2}{\mu^2} + \left(2 + \mu^2\right)^2\frac{p_T^2}{p_L^2}\left(-\frac{p_T^2}{2\,\mu^2\left(1 + \mu^2\right)} + 2\,H_a^2\,\frac{p^2}{p_T^2}\right) - \left(2 + \mu^2\right)H_a^2 \right]\hat{\Phi}$$

$$+ \frac{1}{\mu}\left[\frac{p_T^2\,p^2}{p_L^2} - \mu^2\,\frac{p_T^2}{p_L^2}\left(\frac{p_T^2}{2\left(1 + \mu^2\right)} + 2\left(2 + \mu^2\right)H_a^2\right) \right]\hat{\alpha} + \left(2 + \mu^2\right)H_a\,\frac{p^2}{p_L^2}\,\frac{\hat{\alpha}_0}{\mu} = 0$$

$$(\text{A6})$$

$$\left[\mu^2 + \frac{p_T^2}{p_L^2}\frac{p_T^2 - 4\,\mu^2\left(1+\mu^2\right)H_a^2}{2\,\mu^2\left(1+\mu^2\right)H_a^2}\right]\ddot{\hat{\Psi}} + \left[\mu^2\left(3+2\,\mu^2\right) + \frac{p_T^2}{p_L^2}\left(\frac{1-2\,\mu^2}{\mu^2\left(1+\mu^2\right)}\frac{p_T^2}{2\,H_a^2} - 6\right)\right]H_a\,\dot{\hat{\Psi}} + \mu^2\,p_T^2\,\hat{\Psi}$$

$$+\left[2+\mu^2+\frac{p^2}{p_L^2}\left[-2\left(2+\mu^2\right)+\frac{p_T^2}{\mu^2\,H_a^2}\right] - \frac{p_T^2}{p_L^2}\frac{p_T^2}{2\left(1+\mu^2\right)H_a^2}\right]H_a\,\dot{\hat{\Phi}}$$

$$+\left[\frac{1+\mu^2}{\mu^2}+\frac{\left(2+\mu^2\right)\left(1-2\,\mu^2\right)}{2\,\mu^2\left(1+\mu^2\right)}\frac{p_T^2}{p_L^2}-\left(2+\mu^2\right)\left(\frac{3+2\mu^2}{p_T^2}+\frac{6}{p_L^2}\right)H_a^2\right]p_T^2\,\hat{\Phi}$$

$$-\frac{1}{2\,\mu\,H_a}\frac{p_T^2}{p_L^2}\left(\frac{p_T^2}{1+\mu^2}-4\,\mu^2\,H_a^2\right)\dot{\hat{\alpha}}+\mu\,\frac{p_T^2}{p_L^2}\left(6\,H_a^2-p_L^2-\frac{1-2\,\mu^2}{2\,\mu^2\left(1+\mu^2\right)}p_T^2\right)\hat{\alpha}$$

$$-\left(\frac{p_T^2}{p_L^2}-\mu^2\right)\frac{\dot{\hat{\alpha}}_0}{\mu}-\left(3\,\frac{p_T^2}{p_L^2}-3\,\mu^2-2\,\mu^4\right)H_a\,\frac{\hat{\alpha}_0}{\mu}=0$$

$$\tag{A7}$$

$$-\frac{1}{\mu^2}\left(1-\mu^2\,\frac{p_L^2}{p_T^2}\right)\dot{\hat{\Psi}}-\frac{2+\mu^2}{\mu}\frac{p^2}{p_T^2}H_a\,\hat{\Phi}+\dot{\hat{\alpha}}+2\,H_a\,\hat{\alpha}+\frac{p^2}{p_T^2}\hat{\alpha}_0=0$$

$$\tag{A8}$$

$$\ddot{\hat{\alpha}}+3\,H_a\,\dot{\hat{\alpha}}+\left[p_L^2-\frac{p_T^2}{2\left(1+\mu^2\right)}\right]\hat{\alpha}+\dot{\hat{\alpha}}_0+H_a\,\hat{\alpha}_0$$

$$+\frac{1}{2\,\mu\,H_a}\left(\frac{p_T^2}{1+\mu^2}-4\,\mu^2\,H_a^2\right)\dot{\hat{\Psi}}-\mu\,p_L^2\,\hat{\Psi}+\frac{1}{\mu}\left[p^2-\frac{\mu^2}{2\left(1+\mu^2\right)}p_T^2-2\left(2+\mu^2\right)\mu^2\,H_a^2\right]\hat{\Phi}=0$$

$$\tag{A9}$$

We choose not to eliminate any more modes, but to rewrite these equations as a system of differential equations for $\dot{\hat{\Psi}}$, $\ddot{\hat{\alpha}}$, $\dot{\hat{\Phi}}$ and $\hat{\alpha}_0$. The differential equations for $\hat{\alpha}_0$ and $\hat{\Phi}$ are obtained by differentiating the two equations (A6) and (A8). In solving the numerical system we can replace (A6) and (A8) with their time derivatives, provided that these two equations are imposed as initial conditions (see below). We are thus left with the system of eqs. $\{(A6)^\bullet, (A7), (A8)^\bullet, (A9)\}$, where $(^\bullet)$ indicates that we take the time derivative of the equation. This system can be recast in the form (24), in terms of the coefficients

$$\kappa_{22} = -\frac{1}{\mu}\left(\frac{p_T^2}{p_L^2}-\mu^2\right)$$

$$\kappa_{23} = \mu^2+\frac{p_T^2}{p_L^2}\frac{p_T^2-4\,\mu^2\left(1+\mu^2\right)H_a^2}{2\,\mu^2\left(1+\mu^2\right)H_a^2}$$

$$\kappa_{24} = \left[2+\mu^2+\frac{p^2}{p_L^2}\left[-2\left(2+\mu^2\right)+\frac{p_T^2}{\mu^2\,H_a^2}\right]-\frac{p_T^2}{p_L^2}\frac{p_T^2}{2\left(1+\mu^2\right)H_a^2}\right]H_a$$

$$\kappa_{32} = \frac{p^2}{p_T^2}$$

$$\kappa_{33} = -\frac{1}{\mu}\left(1-\mu^2\,\frac{p_L^2}{p_T^2}\right)$$

$$\kappa_{34} = -\frac{2+\mu^2}{\mu^2}\frac{p^2}{p_T^2}H_a$$

$$\kappa_{42} = \frac{2+\mu^2}{\mu^2}\frac{p^2}{p_L^2}H_a$$

$$\kappa_{43} = \frac{p_T^2}{p_L^2}\frac{1}{H_a}\left[-\frac{p^2}{\mu^2}+\frac{p_T^2}{2\left(1+\mu^2\right)}+\left(2+\mu^2\right)\frac{p_L^2+2\,p_T^2}{p_T^2}H_a^2\right]$$

$$\kappa_{44} = -\frac{2\left(1+\mu^2\right)}{\mu^2}p^2-\frac{2\,p_T^2}{\mu^2}+\left(2+\mu^2\right)^2\frac{p_T^2}{p_L^2}\left(-\frac{p_T^2}{2\,\mu^2\left(1+\mu^2\right)}+2\,H_a^2\frac{p^2}{p_T^2}\right)-\left(2+\mu^2\right)H_a^2$$

$$\tag{A10}$$

and

$$
\begin{aligned}
f_1 =\ & -3\,H_a\,\dot{\hat{\alpha}} - \left[p_L^2 - \frac{p_T^2}{2\,(1+\mu^2)} \right] \hat{\alpha} - H_a\,\hat{\alpha}_0 \\
& - \frac{1}{2\,\mu\,H_a} \left(\frac{p_T^2}{1+\mu^2} - 4\,\mu^2\,H_a^2 \right) \dot{\hat{\Psi}} + \mu\,p_L^2\,\hat{\Psi} - \frac{1}{\mu} \left[p^2 - \frac{\mu^2}{2\,(1+\mu^2)}\,p_T^2 - 2\,(2+\mu^2)\,\mu^2\,H_a^2 \right] \hat{\Phi}
\end{aligned}
$$

$$
\begin{aligned}
f_2 =\ & -\left[\mu^2\,(3+2\,\mu^2) + \frac{p_T^2}{p_L^2} \left(\frac{1-2\,\mu^2}{\mu^2\,(1+\mu^2)}\,\frac{p_T^2}{2\,H_a^2} - 6 \right) \right] H_a\,\dot{\hat{\Psi}} - \mu^2\,p_T^2\,\hat{\Psi} \\
& - \left[\frac{1+\mu^2}{\mu^2} + \frac{(2+\mu^2)\,(1-2\,\mu^2)}{2\,\mu^2\,(1+\mu^2)}\,\frac{p_T^2}{p_L^2} - (2+\mu^2)\,\left(\frac{3+2\mu^2}{p_T^2} + \frac{6}{p_L^2} \right) H_a^2 \right] p_T^2\,\hat{\Phi} \\
& + \frac{1}{2\,\mu\,H_a}\,\frac{p_T^2}{p_L^2} \left(\frac{p_T^2}{1+\mu^2} - 4\,\mu^2\,H_a^2 \right) \dot{\hat{\alpha}} - \mu\,\frac{p_T^2}{p_L^2} \left(6\,H_a^2 - p_L^2 - \frac{1-2\,\mu^2}{2\,\mu^2\,(1+\mu^2)}\,p_T^2 \right) \hat{\alpha} \\
& + \left(3\,\frac{p_T^2}{p_L^2} - 3\,\mu^2 - 2\,\mu^4 \right) H_a\,\frac{\hat{\alpha}_0}{\mu}
\end{aligned}
$$

$$
f_3 = -2\,H_a\,\dot{\hat{\alpha}} - 2\,\mu^2\,H_a\,\frac{p_L^2}{p_T^2}\,\hat{\alpha}_0 - 2\,\mu^3\,H_a\,\frac{p_L^2}{p_T^2}\,\dot{\hat{\Psi}} + 2\,(2+\mu^2)\,\mu\,H_a^2\,\frac{p_L^2}{p_T^2}\,\hat{\Phi}
$$

$$
\begin{aligned}
f_4 =\ & -\frac{1}{\mu}\,\frac{p_T^2}{p_L^2} \left[p^2 - \mu^2 \left(\frac{p_T^2}{2\,(1+\mu^2)} + 2\,(2+\mu^2)\,H_a^2 \right) \right] \dot{\hat{\alpha}} \\
& - \frac{p_T^2}{p_L^2} \left[3\,p^2 - \frac{\mu^2\,p_T^2}{1+\mu^2} + \frac{2\,p^2}{\mu^2} - 4\,(2+\mu^2)\,\mu^2\,H_a^2 \right] \dot{\hat{\Psi}} + 2\,(2+\mu^2)\,\mu\,H_a^2\,\frac{p_T^2}{p_L^2}\,\hat{\alpha}_0 \\
& + \frac{H_a\,p_T^2}{p_L^2} \left[\frac{2}{\mu}\,(1+\mu^2)\,p^2 + \frac{\mu}{1+\mu^2}\,p_T^2 - 4\,(2+\mu^2)\,\mu^3\,H_a^2 \right] \hat{\alpha} + 2\,(1+\mu^2)\,H_a\,p_T^2\,\hat{\Psi} \\
& - \frac{p_T^2}{p_L^2}\,\frac{H_a\,(1+\mu^2)}{\mu^2} \left\{ \frac{1}{p_T^2} \left[2\,p_L^2 + (2+\mu^2)\,p_T^2 \right]^2 - \frac{\mu^4\,(2+\mu^2)^2}{(1+\mu^2)^2} \left[p_T^2 + 4\,(1+\mu^2)\,H_a^2 \right] \right\} \hat{\Phi}
\end{aligned}
$$

$$\tag{A11}$$

As we already mentioned, we need to impose Eqs. (A6) and (A8) as initial condition (this allowed us to replace these two equations with their time derivatives). By imposing them, we obtain the initial conditions for the nondynamical modes in terms of the dynamical ones:

$$
\begin{aligned}
\hat{\alpha}_{0,in} =\ & \frac{1}{p^2\,\mathcal{D}_{in}} \Bigg\{ - \left[4\,(1+\mu^2)\,p_L^2 + (2+\mu^2)^2\,p_T^2 \left(\frac{p_T^2}{1+\mu^2} - 4\,\mu^2\,H_a^2 \right) \right. \\
& \hspace{4cm} \left. + 2\,(2+\mu^2)\,p_L^2\,(2\,p_T^2 - (3+2\,\mu^2)\,\mu^2\,H_a^2) \right] \left(\dot{\hat{\alpha}} + H_a\,\hat{\alpha} \right) \\
& - \mu\,p_L^2 \left[2\,p^2 - \frac{\mu^2}{1+\mu^2}\,p_T^2 + 2\,(2+\mu^2)\,H_a^2 \right] \left[\mu\,H_a\,\hat{\alpha} + (1+\mu^2) \left(2\,\frac{p^2}{p_T^2} - 1 \right) \dot{\hat{\Psi}} \right] \\
& + 2\,\mu\,(2+\mu^2)\,H_a\,p_L^2\,p^2 \left[\frac{2\,H_a\,(2+\mu^2)\,(1+\mu^2)}{p_T^2}\,\dot{\hat{\Psi}} + \hat{\Psi} \right] \Bigg\}_{in}
\end{aligned}
$$

$$
\begin{aligned}
\hat{\Phi}_{in} =\ & \frac{1}{\mathcal{D}_{in}} \Bigg\{ \left[2\,p^2 - \frac{\mu^2}{1+\mu^2}\,p_T^2 - 2\,H_a^2\,(2+\mu^2)\,(1+2\,\mu^2) \right] \left(-\frac{\dot{\hat{\Psi}}}{H_a} + \mu\,\hat{\alpha} \right) \\
& + 2\,\mu^2\,p_L^2\,\hat{\Psi} - 2\,\mu\,(2+\mu^2)\,H_a \left(\dot{\hat{\alpha}} + H_a\,\hat{\alpha} \right) \Bigg\}_{in}
\end{aligned}
$$

$$
\mathcal{D} \equiv \frac{1+\mu^2}{p_T^2} \left\{ \left(2\,p^2 - \frac{\mu^2}{1+\mu^2}\,p_T^2 \right)^2 - 2\,H_a^2\,(2+\mu^2) \left[2\,(1+\mu^2)\,p^2 + \frac{\mu^2}{1+\mu^2}\,p_T^2 \right] \right\} \tag{A12}
$$

The initial conditions for $\hat{\Psi}$, $\dot{\hat{\Psi}}$, $\hat{\alpha}$, $\dot{\hat{\alpha}}$ are obtained from the early time expansion of the action (30). Namely, we

need to expand at early times the three matrices K, X, Ω^2 entering in this action. [16] We find

$$S_{2\mathrm{dS}} = \frac{1}{2} \int d^3k\, dt\, \left\{ |\dot{H}_+|^2 - p^2\, |H_+|^2 + |\dot{\Delta}_+|^2 - p^2\, |\Delta_+|^2 + \frac{\mu\, H_a}{\sqrt{2}} \frac{p_T}{p} \left(\dot{H}_+^*\, \Delta_+ - \dot{\Delta}_+\, H_+^* + \mathrm{h.c.} \right) \right\} \tag{A13}$$

where

$$\hat{\Psi} \equiv \frac{1}{\sqrt{a}\,b} \frac{2p^2 + \mu^2\,\left(p^2 + p_L^2\right)}{\sqrt{2}\,\left(1 + \mu^2\right)\,p_T^2}\, H_+ \quad , \quad \hat{\alpha} \equiv \frac{1}{\sqrt{a}\,b} \left[-\frac{p}{p_T}\, \Delta_+ + \frac{\mu\,\left(p^2 + p_L^2\right)}{\sqrt{2}\,p_T^2}\, H_+ \right] \tag{A14}$$

The next to leading order corrections to (A13) and (A14) are of $\mathrm{O}\left(H_a^2/p^2\right)$ (either p_L or p_T). Not surprisingly, the early time frequency is controlled by the momentum term (which is a common result in the theory of cosmological perturbations on a inflationary background). What is nontrivial is the relation (A14) between the canonical variables and the original modes, which, as we shall now see, is needed in order to set the initial conditions for the latter.

Since the early time frequency varies adiabatically as

$$\omega^2 \approx p^2 \quad , \quad \frac{\dot{\omega}}{\omega^2} \approx -\frac{p^2 + \mu^2\, p_T^2}{p^2} \frac{H_a}{p} \ll 1$$

we can set the adiabatic initial conditions for the canonical variables [17]

$$H_{+,in} = \Delta_{+,in} = \frac{1}{\sqrt{2p}} \quad , \quad \dot{H}_{+,in} = -i\, p\, H_{+,in} \quad , \quad \dot{\Delta}_{+,in} = -i\, p\, \Delta_{+,in} \tag{A15}$$

which are $\mathrm{O}\left(H_a/p\right)$ accurate. Using (A14) we then obtain the initial conditions for the original modes as

$$\begin{aligned}
\hat{\Psi}_{in} &= \frac{1}{\sqrt{a}\,b} \frac{2p^2 + \left(p^2 + p_L^2\right)\,\mu^2}{2\sqrt{p}\,p_T^2\,\left(1 + \mu^2\right)} \left[1 + \mathrm{O}\left(\frac{H_a^2}{p^2}\right) \right] \quad , \quad \dot{\hat{\Psi}}_{in} = -i\, p\, \hat{\Psi}_{in} \left[1 + \mathrm{O}\left(\frac{H_a}{p}\right) \right] \\
\hat{\alpha}_{in} &= -\frac{1}{\sqrt{a}\,b} \frac{\sqrt{2}\,p\,p_T - \left(p^2 + p_L^2\right)\,\mu}{2\sqrt{p}\,p_T^2} \left[1 + \mathrm{O}\left(\frac{H_a^2}{p^2}\right) \right] \quad , \quad \dot{\hat{\alpha}}_{in} = -i\, p\, \hat{\alpha}_{in} \left[1 + \mathrm{O}\left(\frac{H_a}{p}\right) \right]
\end{aligned} \tag{A16}$$

The initial conditions for the nondynamical modes are then obtained by inserting these expressions into (A12). These are the initial conditions for the numerical evolution discussed in the main text, which leads to the solutions given in Figure 1.

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[1] Hinshaw, G., et al. 2008, arXiv:0803.0732
[2] C. L. Bennett *et al.*, Astrophys. J. **464**, L1 (1996) [arXiv:astro-ph/9601067].
[3] D. N. Spergel *et al.* [WMAP Collaboration], Astrophys. J. Suppl. **148**, 175 (2003) [arXiv:astro-ph/0302209].
[4] A. de Oliveira-Costa, M. Tegmark, M. Zaldarriaga and A. Hamilton, Phys. Rev. D **69**, 063516 (2004) [arXiv:astro-ph/0307282].; G. Efstathiou, Mon. Not. Roy. Astron. Soc. **348**, 885 (2004) [arXiv:astro-ph/0310207]; C. Copi, D. Huterer, D. Schwarz and G. Starkman, Phys. Rev. D **75**, 023507 (2007) [arXiv:astro-ph/0605135].
[5] K. Land and J. Magueijo, Phys. Rev. Lett. **95**, 071301 (2005) [arXiv:astro-ph/0502237].; T. R. Jaffe, A. J. Banday, H. K. Eriksen, K. M. Gorski and F. K. Hansen, Astrophys. J. **629**, L1 (2005) [arXiv:astro-ph/0503213].;
[6] F. K. Hansen, P. Cabella, D. Marinucci and N. Vittorio, Astrophys. J. **607**, L67 (2004) [arXiv:astro-ph/0402396].; H. K. Eriksen, F. K. Hansen, A. J. Banday, K. M. Gorski and P. B. Lilje, Astrophys. J. **605**, 14 (2004) [Erratum-ibid. **609**, 1198 (2004)] [arXiv:astro-ph/0307507].; F. K. Hansen, A. J. Banday and K. M. Gorski, arXiv:astro-ph/0404206.

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[16] The only time dependent quantities entering in these expressions are the scale factors. Some care needs to be taken in the expansion, since $H_b > H_a$, so that $p_T \gg p_L$ in the asymptotic past. The expansion (A13) is valid as long as $H_b < 2\,H_a$. If this is not the case, a term proportional to p_L^2 becomes smaller than a term proportional to p_T at asymptotically early times, and the expansion in powers of (H/p) that we have performed (where p is either of p_L or p_T) becomes invalid. The condition $H_b < 2\,H_a$ is not particularly stringent, since such a large anisotropic is certainly incompatible with the observations.

[17] This is precisely what is done in the standard case [30], where the canonical variable for the scalar perturbation is the Mukhanov-Sasaki [38] variable v.

[7] C. J. Copi, D. Huterer, D. J. Schwarz and G. D. Starkman, arXiv:0808.3767 [astro-ph].

[8] N. E. Groeneboom and H. K. Eriksen, arXiv:0807.2242 [astro-ph].

[9] C. Armendariz-Picon and L. Pekowsky, arXiv:0807.2687 [astro-ph].

[10] A. E. Gumrukcuoglu, C. R. Contaldi and M. Peloso, arXiv:astro-ph/0608405, Proceeding of the "11th Marcel Grossmann Meeting On General Relativity" Ed. H. Kleinert, R.T. Jantzen and R. Ruffini. Hackensack, World Scientific, 2008.

[11] A. E. Gumrukcuoglu, C. R. Contaldi and M. Peloso, JCAP **0711**, 005 (2007) [arXiv:0707.4179 [astro-ph]].

[12] T. S. Pereira, C. Pitrou and J. P. Uzan, JCAP **0709**, 006 (2007) [arXiv:0707.0736 [astro-ph]].

[13] C. Pitrou, T. S. Pereira and J. P. Uzan, JCAP **0804**, 004 (2008) [arXiv:0801.3596 [astro-ph]].

[14] A. E. Gumrukcuoglu, L. Kofman and M. Peloso, arXiv:0807.1335 [astro-ph].

[15] R. W. Wald, Phys. Rev. D **28**, 2118 (1983).

[16] J. D. Barrow and S. Hervik, Phys. Rev. D **73**, 023007 (2006) [arXiv:gr-qc/0511127]; Phys. Rev. D **74**, 124017 (2006) [arXiv:gr-qc/0610013].

[17] N. Kaloper, Phys. Rev. D **44**, 2380 (1991). E. Di Grezia, G. Esposito, A. Funel, G. Mangano and G. Miele, Phys. Rev. D **68**, 105012 (2003) [arXiv:gr-qc/0305050].

[18] L. H. Ford, Phys. Rev. D **40**, 967 (1989).

[19] C. Armendariz-Picon, JCAP **0407**, 007 (2004) [arXiv:astro-ph/0405267]. T. Koivisto and D. F. Mota, Astrophys. J. **679**, 1 (2008) [arXiv:0707.0279 [astro-ph]]; T. S. Koivisto and D. F. Mota, JCAP **0808**, 021 (2008) [arXiv:0805.4229 [astro-ph]].

[20] D. J. Schwarz and B. Weinhorst, arXiv:0706.0165 [astro-ph]. ; M. Blomqvist, E. Mortsell and S. Nobili, JCAP **0806**, 027 (2008)[arXiv:0806.0496 [astro-ph]].

[21] L. Ackerman, S. M. Carroll and M. B. Wise, Phys. Rev. D **75**, 083502 (2007) [arXiv:astro-ph/0701357].

[22] G. Dvali, A. Gruzinov and M. Zaldarriaga, Phys. Rev. D **69**, 023505 (2004) [arXiv:astro-ph/0303591]; L. Kofman, arXiv:astro-ph/0303614.

[23] T. R. Dulaney, M. I. Gresham and M. B. Wise, Phys. Rev. D **77**, 083510 (2008) [arXiv:0801.2950 [astro-ph]].

[24] T. R. Dulaney and M. I. Gresham, arXiv:0805.1078 [gr-qc].

[25] M. A. Clayton, arXiv:gr-qc/0104103.

[26] K. Dimopoulos, D. H. Lyth and Y. Rodriguez, arXiv:0809.1055 [astro-ph].;

[27] A. Golovnev, V. Mukhanov and V. Vanchurin, JCAP **0806**, 009 (2008) [arXiv:0802.2068 [astro-ph]].

[28] S. Kanno, M. Kimura, J. Soda and S. Yokoyama, arXiv:0806.2422 [hep-ph].

[29] B. Himmetoglu, C. R. Contaldi and M. Peloso, arXiv:0809.2779 [astro-ph].

[30] V. F. Mukhanov, H. A. Feldman and R. H. Brandenberger, Phys. Rept. **215**, 203 (1992).

[31] S. M. Carroll and E. A. Lim, Phys. Rev. D **70**, 123525 (2004) [arXiv:hep-th/0407149].

[32] R. Arnowitt, S. Deser and C. W. Misner, *Gravitation: an introduction to current research*, Louis Witten ed. (Wiley 1962), chapter 7, pp 227-265 [gr-qc/0405109].

[33] K. Bamba and S. Nojiri, arXiv:0811.0150 [hep-th].

[34] S. Yokoyama and J. Soda, JCAP **0808**, 005 (2008) [arXiv:0805.4265 [astro-ph]].

[35] T. Koivisto and D. F. Mota, JCAP **0806**, 018 (2008) [arXiv:0801.3676 [astro-ph]].

[36] H. C. Cheng, M. A. Luty, S. Mukohyama and J. Thaler, JHEP **0605**, 076 (2006) [arXiv:hep-th/0603010].

[37] N. Arkani-Hamed, H. C. Cheng, M. A. Luty and S. Mukohyama, JHEP **0405**, 074 (2004) [arXiv:hep-th/0312099].

[38] V. F. Mukhanov, JETP Lett. **41**, 493 (1985) [Pisma Zh. Eksp. Teor. Fiz. **41**, 402 (1985)]; M. Sasaki, Prog. Theor. Phys. **76**, 1036 (1986).